A 3-local Characterization of the Thompson Sporadic Simple Group

by

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Abstract

In this thesis we characterize the Thompson sporadic simple group by its 3-local structure. We study a faithful completion, G, of an amalgam of type F_3 with the property that $N_G(Z(L_\beta)) = G_\beta$. We first assume no additional 3-local structure and use a $\mathcal{K}$-proper hypothesis to establish that the completion G with this property contains a subgroup Y of order 3 such that $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$. Secondly, we assume that G contains such a subgroup Y with $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$ and show that for an involution $t \in G$, $C_G(t)$ has shape $2_+^{1+8}.\mathrm{Alt}(9)$. We then invoke a theorem of Parrott to show that $G \cong \mathrm{Th}$.

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Contents

Introduction

The Classification of Finite Simple Groups was announced in the early 1980's and it stated that a finite simple group is isomorphic to one of the following.

(i) A cyclic group of prime order.

(ii) An alternating group of degree at least five.

(iii) A finite simple group of Lie type.

(iv) One of 26 sporadic finite simple groups.

The proof runs over $10,000$ to $15,000$ pages, some in unpublished papers, and as such is inefficient and difficult to understand. As a result of this, soon after the classification was announced work began by Gorenstein, Lyons and Solomon and continued by Korchagina (see [15], [16], [17], [18], [19], [20], [22], [23], [24] and [26]) in what has become known as the second generation proof. This hopes to provide a proof of the classification that is more accessible.

In addition to this second generation proof, work has begun to classify the finite simple groups using the amalgam method. This method focusses on the group theoretic structure of the groups in the amalgam, rather than the completions of the amalgams. This aims to classify simple groups of local characteristic p for an arbitrary prime p (see [28] for an overview, [27] and [30]). Work using this method has come to be known as the third generation "proof". An important stage in this work is to recognise the groups themselves, given the amalgams of groups (see [25], [33], [34] and [29] for example) and this work forms part of this stage.

We recall that given P, any non-trivial p-subgroup of a group G for p a prime, then H is said to be a p-local subgroup of G if $H = N_G(P)$. Also, a group G is p-constrained if $C_G(O_p(G)) \leq O_p(G)$ and a group is said to be of local characteristic p if $N_G(P)$ is p-constrained for all non-trivial p-subgroups, P, of G.

Suppose that G is a group for which all the composition factors of G are known simple groups. Then G is said to be a $\mathcal{K}$-group. A group is said to be $\mathcal{K}$-proper if every proper subgroup of G is a $\mathcal{K}$-group.

Throughout this thesis we mainly use Atlas [8] notation for groups and conjugacy classes. In particular we use Atlas notation for group extensions, see [8, page xx]. However, for $n \in \mathbb{N}$, we use $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ to denote the symmetric and alternating groups of degree n and $\mathrm{Dih}(n)$ and $\mathrm{SDih}(n)$ to denote the dihedral and semi-dihedral group of order n respectively. We also let Q_8 be the quaternion group of order 8, namely the group $\mathrm{Q}_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, xyx = y \rangle$. We say that G has shape $G_1.G_2.\ldots.G_n$, denoted $G \sim G_1.G_2.\ldots.G_n$, if G has a normal series with factors of shape G_i. Throughout this work we reserve the notation $H = (3 \times \mathrm{G}_2(3)) : 2$ to denote the group H such that H has a normal subgroup of index 2 which is isomorphic to $3 \times \mathrm{G}_2(3)$ and a subgroup A of order 2 such that A inverts the normal subgroup of order 3 and acts as the outer automorphism on $\mathrm{G}_2(3)$.

Throughout this work we require the following hypothesis.

Hypothesis A *Let G be a finite group and $S \in \mathrm{Syl}_3(G)$. Suppose that:*

(i) $Z_\beta = Z(S)$ has order 3 and $Z_\alpha = Z_2(S)$ has order 9;

(ii) $G_\alpha = N_G(Z_\alpha) \sim 3^{2+3+2+2} : 2.\mathrm{Sym}(4)$ is 3-constrained;

(iii) $G_\beta = N_G(Z_\beta) \sim 3^{1+2+1+2+1+2} : 2.\mathrm{Sym}(4)$ *is* 3*-constrained; and*

(iv) $O_3(\langle G_\alpha, G_\beta\rangle) = 1$.

We note that a group G which satisfies the above hypotheses can also be seen as the completion of an amalgam of type F_3, $\mathcal{F}_3 = \mathcal{F}_3(G_\alpha, G_\beta, G_{\alpha\beta})$, such that $N_G(Z_\beta) = G_\beta$. This is shown in work on weak BN-pairs by Delgado and Stellmacher, [9]. Throughout this thesis we work predominantly with the amalgam and its completion rather than the hypothesis as it is stated above. We prove two main theorems and a corollary under the assumptions of Hypothesis A. These are as follows.

Theorem A *Suppose that G is a $\mathcal{K}$-proper group and that $S \in \mathrm{Syl}_3(G)$ such that G and S satisfy Hypothesis A. Then there exists a subgroup $Y \leq G$, such that $|Y| = 3$ and* $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$.

Theorem B *Let G and S satisfy Hypothesis A. Suppose that Y is the subgroup of order* 3 *in Theorem A such that $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$. Then there exists an involution $t \in G$ such that $G \neq C_G(t)O_{2'}(G)$ and $C_G(t)$ satisfies:*

(i) $R = O_2(C_G(t))$ *is extra-special of order* 2^9*; and*

(ii) $C_G(t)/R \cong \mathrm{Alt}(9)$.

Corollary *Suppose G, S and Y satisfy Theorem B. Then $G \cong \mathrm{Th}$.*

Together these theorems and corollary characterize the Thompson sporadic simple group Th, otherwise known as F_3. The corollary follows from Theorem B by invoking a result of Parrott, [35], see Theorem 1.1.23.

Chapter 1 contains a number of preliminary definitions and results including the introduction of a rank 2 amalgam of finite groups and its associated coset graph. We then go on to define the weak BN-pairs of characteristic p, as introduced and classified in [9] and further investigated in [32]. In addition to this we define a concept of p-generated amalgams. The recognition result in Chapter 2 is used in Chapter 5 in order to help us prove Theorem A. This chapter relies heavily on a $\mathcal{K}$-proper group hypothesis and is the only place in which this hypothesis is used, apart from the use of the Atlas [8] to recognise the maximal subgroup of $\mathrm{GO}_8^+(2)$ in Chapter 6. In Chapter 3 we prove three technical theorems that are applied in Chapter 6 to prove Theorem B. In order to prove these theorems we first define some notation for the two isomorphism types of $3^3 : \mathrm{Sym}(4)$ in $\mathrm{Sym}(9)$. We also discuss certain $\mathrm{Alt}(9)$ and $3^3 : \mathrm{Sym}(4)$ modules of dimension 8 over $GF(2)$. Chapter 4 contains results concerning the structure of amalgams of types $\mathrm{G}_2(3)$ and F_3, two of the types of weak BN-pairs. Included in this chapter are some of the results of Parker and Rowley from [32, Section 13]. Chapter 5 contains a number of results about the coset graph of an amalgam of type F_3. We go on to complete the proof of Theorem A. The subgroup structure of an amalgam of type F_3 is further investigated in Chapter 6. This chapter concludes with a proof of Theorem B.

0.1 Overview of the Proofs

We now give an overview of the method used to prove Theorems A and B.

Let G be a completion of an amalgam of type F_3, $\mathcal{F}_3 = \mathcal{F}_3(G_\alpha, G_\beta, G_{\alpha\beta})$, and $\Gamma = \Gamma(G, G_\alpha, G_\beta, G_{\alpha\beta})$ be its associated right coset graph. Suppose that $S_{\alpha\beta} \in \mathrm{Syl}_3(G_{\alpha\beta})$. Let T be a complement to $S_{\alpha\beta}$ in $G_{\alpha\beta}$ and Θ be the connected subgraph of Γ which is fixed by T and contains the edge $\{\alpha, \beta\}$. Also, let Θ_β be the set of vertices of Θ which are in the same G-orbit as β. The flowchart in Figure 1 shows the method used to prove Theorem A.

Given $\beta \in \Gamma$, let t_β be the unique involution such that $t_\beta Q_\beta \in Z(G_\beta/Q_\beta)$. We note that t_β is conjugate to the elements of T and this is proven in Chapter 5. Also suppose $\beta - 3$ is a vertex in Γ of distance 3 from β. We consider the subgraph of Γ which is fixed

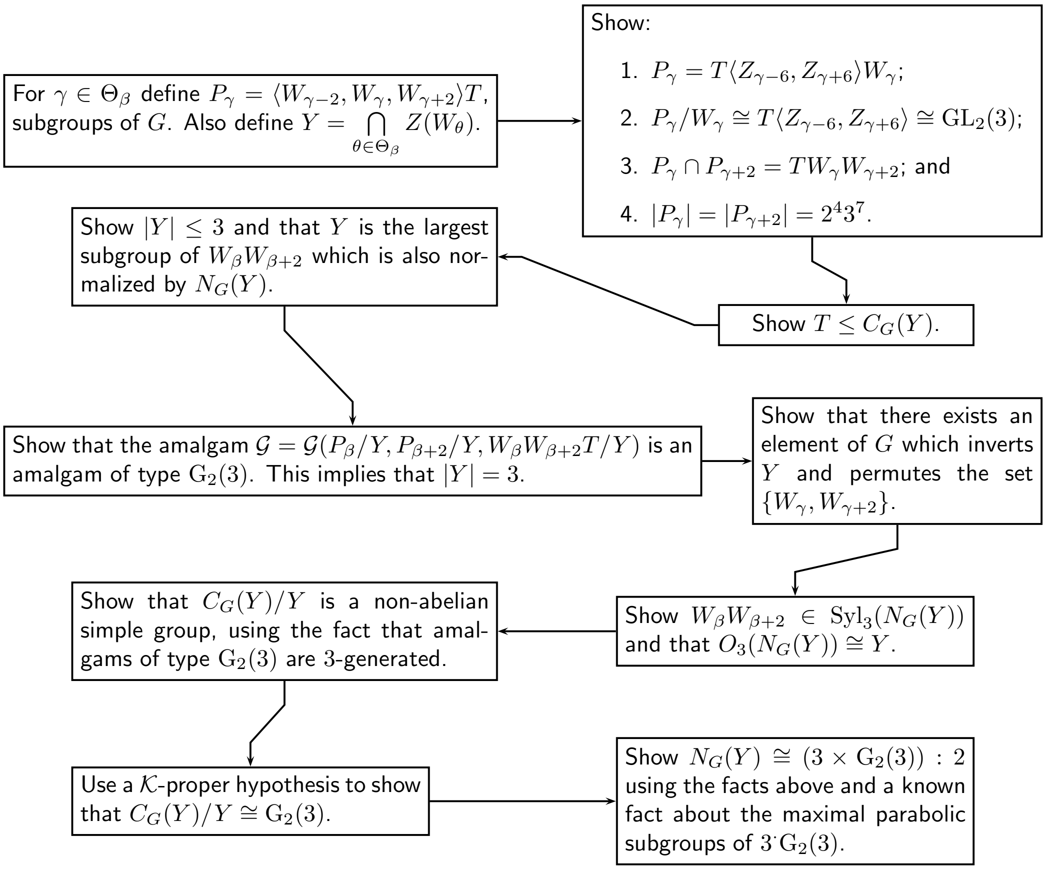

Figure 1: Overview of Proof of Theorem A.

by t_β. Let β, $\beta-6$, $\rho+3$ and $\rho-3$ be the four vertices in Γ of distance 3 from $\beta-3$ shown in Figure 6.1 in Chapter 6. Then these vertices lie on the subgraph of Γ fixed by t_β. For a subgroup X of G we let $\widetilde{X} = C_X(t_\beta)$. The flowchart in Figure 2 shows the method used to prove Theorem B.

We note that we do not need any information about the centralizers in $\widetilde{G}$ of the subgroups of J of order 3 which correspond to the elements of $3^3 : \mathrm{Sym}(4)^+$ conjugate to (123). We also note that we prove equivalent results in the two cases $L/R \cong \mathrm{Alt}(9)$ and $L/R \cong 3^3 : \mathrm{Sym}(4)^+$ in different ways. This is due to the fact that we can obtain different information more easily in each case. In the case when $L/R \cong \mathrm{Alt}(9)$ we are able to obtain all the information about the structure of the Sylow 3-subgroups we need in order to prove that $R/\langle t_\beta \rangle$ is strongly closed in a Sylow 2-subgroup of $\widetilde{G}/\langle t_\beta \rangle$. However, we cannot use the same method to prove the result in the case $L/R \cong 3^3 : \mathrm{Sym}(4)^+$ as there are elements of order 3 that we do not know enough information about. Since we can obtain a large amount of information about the structure of the Sylow 2-subgroups of $3^3 : \mathrm{Sym}(4)^+$, it is this information we use to prove the result in the other case. We now give more details about the proof of these results.

Let H be a group, $F \leq H$ and $N = N_H(F)$. Suppose that $N/F = K$ where $K \cong \mathrm{Alt}(9)$. If gF is in K-class 3A, we define $\mathcal{A} = \{g^H\}$. Similarly, $\mathcal{B} = \{g^H\}$ and $\mathcal{C} = \{g^H\}$ where gF is in K-conjugacy class 3B and 3C respectively. We apply Theorem C in the case $F = R/\langle t_\beta \rangle$ and $N = L/\langle t_\beta \rangle$ in order to show that $R/\langle t_\beta \rangle$ is strongly closed in $L/\langle t_\beta \rangle$ when $L/R \cong \mathrm{Alt}(9)$.

Theorem C *Suppose that H is a group, $F \leq H$ and $N = N_H(F)$. Assume that:*

(i) $N/F = K$, where $K \cong \mathrm{Alt}(9)$, F is elementary abelian of order 2^8 and F is the unique minimal normal subgroup of N;

(ii) the elements of K-conjugacy class 3A act fixed-point-freely on F;

(iii) the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are disjoint; and

(iv) if $g \in \mathcal{B} \cap N$ or $g \in \mathcal{C} \cap N$, then $N_H(\langle g \rangle) \leq N$.

Then F is strongly closed in N with respect to H.

The method used to prove this theorem uses known facts about the structure of the 3-subgroups of Alt(9) and is shown in Figure 3.

We apply Theorem D in the case $F = R/\langle t_\beta \rangle$ and $N = L/\langle t_\beta \rangle$ in order to show that $R/\langle t_\beta \rangle$ is strongly closed in $L/\langle t_\beta \rangle$ when $L/R \cong 3^3 : \mathrm{Sym}(4)^+$.

Theorem D *Suppose that H is a group, $F \leq H$ and $N = N_H(F)$. Assume that:*

(i) $H/F = K$*, where* $K \cong 3^3 : \mathrm{Sym}(4)^+$ *and* F *is elementary abelian of order* 2^8*; and*

(ii) the elements of K*-conjugacy class 3A act fixed-point-freely on* F*;*

Then F *is strongly closed in* L *with respect to* H*.*

The method used to prove this theorem uses known facts about the structure of the 2-subgroups of $3^3 : \mathrm{Sym}(4)^+$ and is shown in Figure 4.

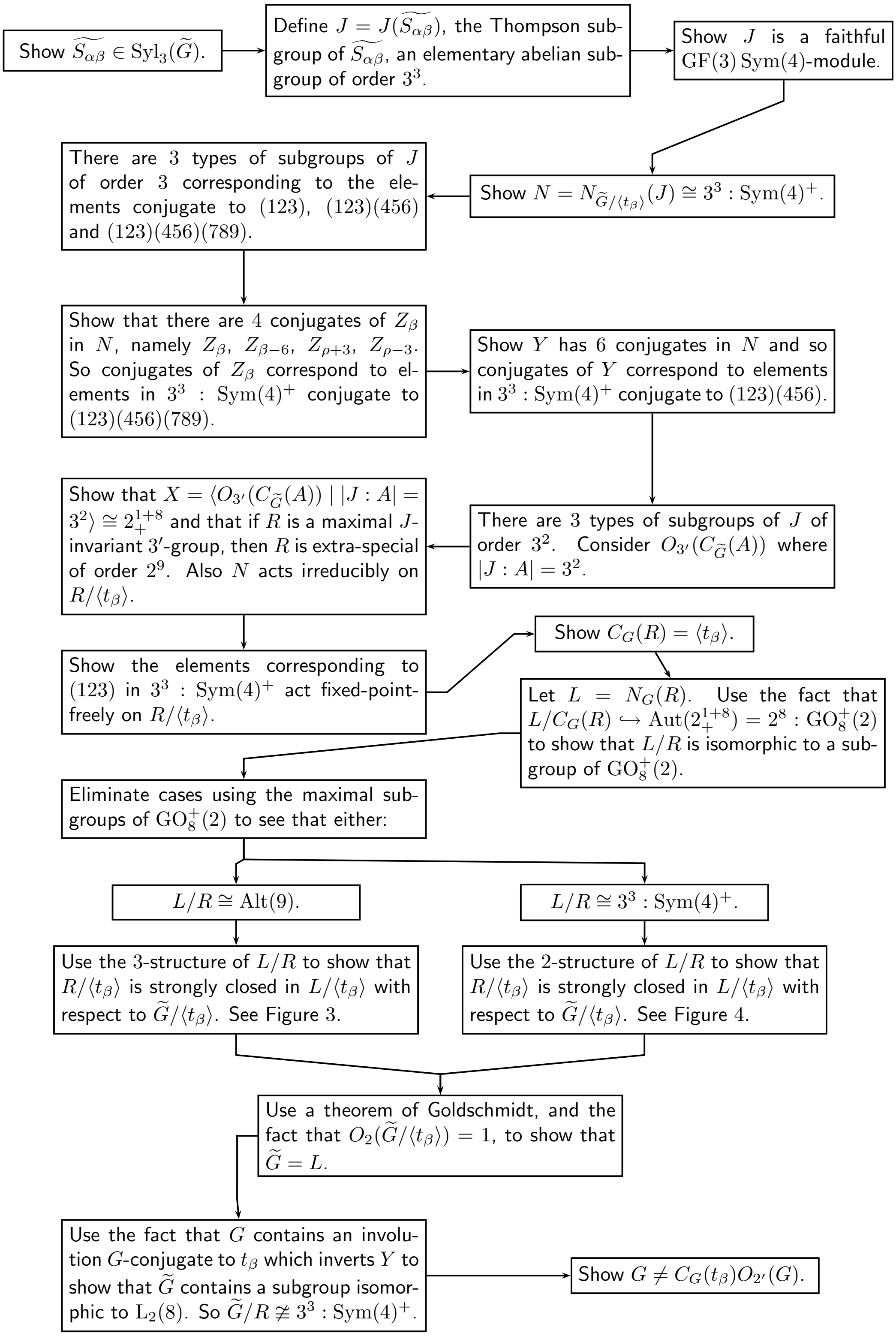

Figure 2: Overview of Proof of Theorem B.

We can consider F as a $\mathrm{GF}(2)K$-module of dimension 3 such that elements in K-conjugacy class act fixed-point-freely on F.

Suppose that F is not strongly closed in N and let $P \in \mathrm{Syl}_2(N)$. Let F^x be a conjugate of F in H such that we may choose $r \in (F^x \cap P)\backslash F$.

Show either $|C_N(r)| = 2^4.2^5.3.5$ or $|C_N(r)| = 2^4.2^6.3$ and that the non-trivial elements of the Sylow 3-subgroups of $C_N(r)$ are in K-conjugacy classes 3A or 3C respectively.

Since elements in K-conjugacy class act fixed-point-freely on F, $C_{N^x}(r)$ does not contain elements from $N^x/F^x \cong K$-conjugacy class 3A. So if $g \in C_{N^x}(r)$ has order 3, then $g \in \mathcal{B}$ or $g \in \mathcal{C}$.

Show $\mathrm{Syl}_3(C_{N^x}(r)) \subseteq \mathrm{Syl}_3(C_H(r))$ using assumption (iv).

So $C_H(r)$ does not contain any elements from K-conjugacy class 3A. Therefore $C_N(r)$ does not contain any elements from K-conjugacy class 3A.

Show $\mathrm{Syl}_3(C_N(r)) \subseteq \mathrm{Syl}_3(C_H(r))$. This implies that the Sylow 3-subgroups of $C_N(r)$ and $C_{N^x}(r)$ are H-conjugate.

Consider the orders of $C_{C_N(r)}(D)$, $C_{C_{N^x}(r)}(D_1)$ and $C_{C_H(r)}(D)$ where $D \in \mathrm{Syl}_3(C_N(r))$ and $D_1 \in \mathrm{Syl}_3(C_{N^x}(r))$ to derive a contradiction.

Figure 3: Case when L/R is isomorphic to $\mathrm{Alt}(9)$.

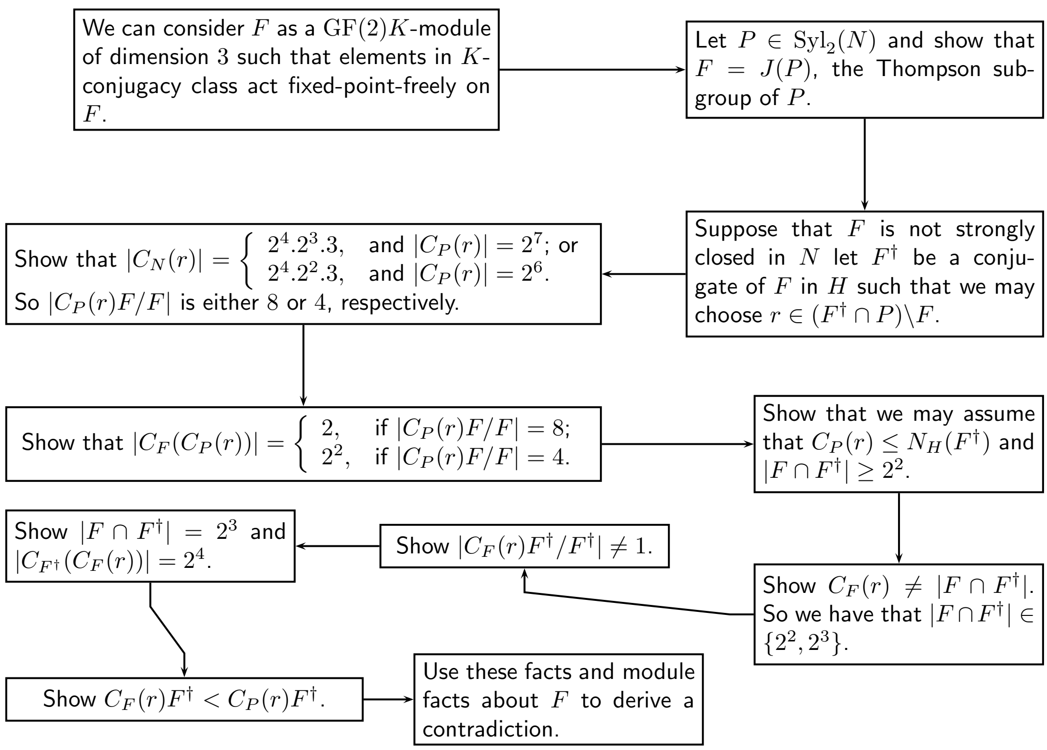

Figure 4: Case when L/R is isomorphic to $3^3 : \mathrm{Sym}(4)^+$.

Chapter 1

Preliminaries

This chapter contains a number of preliminary results and definitions that we shall require throughout this thesis.

1.1 Elementary Results and Definitions

We first state a number of standard results.

Lemma 1.1.1 *(Frattini Lemma) Suppose that* $H \trianglelefteq G$ *and* $P \in \mathrm{Syl}_p(H)$*, for a prime* p*. Then* $G = N_G(P)H$*.*

Proof. See [36, Lemma 5.13]. □

Lemma 1.1.2 *(Dedekind's Modular Law) Suppose that* A*,* B *and* C *are subgroups of a group* G *such that* $B \leq C$*. Then*

$$AB \cap C = (A \cap C)B.$$

Proof. See [36, 7.3]. □

Lemma 1.1.3 *(Three Subgroup Lemma) Suppose that* G *is a group and* H*,* K *and* L *are subgroups of* G*. If* $[H, K, L] = [K, L, H] = 1$*, then* $[L, H, K] = 1$*.*

Proof. See [13, Theorem 2.2.3]. □

Lemma 1.1.4 *(Coprime Action) Suppose that G, N and A are groups such that $|G|$ and $|A|$ are coprime, $A \leq \mathrm{Aut}(G)$ and N is an A-invariant normal subgroup of G. Then:*

(i) $G = C_G(A)[G, A]$;

(ii) *if G is abelian then* $G = C_G(A) \times [G, A]$;

(iii) $[G, A, A] = [G, A]$;

(iv) *if A is an elementary abelian p-group, for p a prime, and $|A| \geq p^2$, then*

$$G = \langle C_G(A_i) \mid |A : A_i| = p \rangle = \langle C_G(a) \mid a \in A^{\#} \rangle;$$

(v) *G has an A-invariant Sylow p-subgroup for any prime p; and*

(vi) $C_{G/N}(A) = C_G(A)N/N$.

Proof. See [13, 5.2.3, 5.3.5, 5.3.6, 5.3.16] for parts (i) to (iv) and [2, 18.7 (1) and (4)] for parts (v) and (vi). □

Lemma 1.1.5 *Suppose that p and r are distinct primes. Let A be a non-trivial elementary abelian p-group and V a faithful $\mathrm{GF}(r)A$-module. If $C_V(A) = \{0\}$, then*

$$V = \bigoplus_{A_i \in \mathcal{M}} C_V(A_i),$$

where $\mathcal{M}$ is the set of subgroups of index p in A.

Proof. (See [1, 2.1]) By Coprime Action we have, $V = \langle C_V(A_i) \mid |A : A_i| = p \rangle$. Suppose that $B \in \mathcal{M}$. Then $|A : B| = p$ and $V = [V, B] \oplus C_V(B)$, again by Coprime Action. Let

$C \in \mathcal{M}\backslash\{B\}$. Then we have that $A = CB$ and therefore

$$\{0\} = C_V(A) = C_V(CB) = C_V(C) \cap C_V(B).$$

So $C_V(C) = [C_V(C), B] \leq [V, B]$ and the result follows. □

Lemma 1.1.6 *(Thompson's $A \times B$ Lemma) Let P be a p-group, for a p prime. Suppose AB is a group which acts on P such that $[A, B] = 1$, B is a p-group and A a p'-group. If $[C_P(B), A] = 1$, then $[P, A] = 1$.*

Proof. See [2, 24.2]. □

The following definition will be required in Chapter 4.

Definition 1.1.7 We define the second centre of a group G to be the subgroup $Z_2(G)$ of G that contains $Z(G)$ such that $Z_2(G)/Z(G) = Z(G/Z(G))$.

We now give a number of results concerning p-groups, for p a prime. We first require two definitions.

Definition 1.1.8 Let P be a p-group, for p a prime. Then $\Omega_1(P) = \langle x \in P \mid x^p = 1\rangle$.

Definition 1.1.9 Let P be a p-group, for a prime p, and let $\mathcal{A}(P)$ denote the set of abelian subgroups of P of maximal order. We define the Thompson subgroup of P, denoted by $J(P)$, to be

$$J(P) = \langle A \mid A \in \mathcal{A}(P)\rangle.$$

Lemma 1.1.10 *Suppose that $S \in \mathrm{Syl}_p(G)$ for some finite group G and prime p. Let R be a p-group that contains $J(S)$. Then $J(R) = J(S)$.*

Proof. See [13, Lemma 8.2.2,(i)]. □

The following lemma shows that the normalizer of the Thompson subgroup of a Sylow p-subgroup of a finite group G recognises some of the conjugacy classes of the group.

Lemma 1.1.11 *Let $S \in \mathrm{Syl}_p(G)$ for some finite group G and prime p. Suppose that $x, y \in Z(J(S))$. Then $x^g = y$, for some $g \in G$ if and only if $x^n = y$, for some $n \in N_G(J(S))$.*

Proof. Clearly, if x is $N_G(J(S))$-conjugate to y, then x is G-conjugate to y since $N_G(J(S)) \leq G$. Suppose that $x^g = y$ for some $g \in G$. Since $x \in Z(J(S))$, $J(S) \leq C_G(x)$. Hence we can choose $T \in \mathrm{Syl}_p(C_G(x))$ such that $J(S) \leq T$. Similarly, we can choose $R \in \mathrm{Syl}_p(C_G(y))$ such that $J(S) \leq R$. We have that $T^g \leq (C_G(x))^g = C_G(x^g) = C_G(y)$ and hence $T^g \in \mathrm{Syl}_p(C_G(y))$. Therefore, there exists $k \in C_G(y)$ such that $T^{gk} = R$.

So, using Lemma 1.1.10, we have that

$$J(S)^{gk} = J(T)^{gk} = J(T^{gk}) = J(R) = J(S).$$

Hence $gk \in N_G(J(S))$ and $x^{gk} = y^k = y$. $\square$

Lemma 1.1.12 *Let P be a p-group, for p a prime and H be a non-trivial normal subgroup of P. Then $Z(P) \cap H \neq 1$.*

Proof. See [36, Corollary 5.8]. $\square$

Definition 1.1.13 Suppose that P is a p-subgroup of the group G for p a prime. If $[P, G] = 1$, then P is said to be a p-central subgroup of G.

Corollary 1.1.14 *Let P be a p-group and $A \trianglelefteq P$ with $|A| = p$. Then $A \leq Z(P)$.*

Proof. This follows from Lemma 1.1.12 with $H = A$. $\square$

Definition 1.1.15 Let P be a Sylow p-subgroup of a group G. If $G = PO_{p'}(G)$, then G is said to have a normal p-complement.

The next lemma is Burnside's normal p-complement theorem.

Lemma 1.1.16 *Suppose G is a group and $S \in \mathrm{Syl}_p(G)$ for some prime p. If $S \leq Z(N_G(S))$, then G has a normal p-complement.*

Proof. See [13, Theorem 7.4.3]. $\square$

We require the following result in Chapter 6.

Lemma 1.1.17 *Let G be a finite group and assume that G acts transitively on a set Ω. Let $\omega \in \Omega$ and $K = \mathrm{Stab}_G(\omega)$. If $H \leq G$ such that $G = KH$, then H acts transitively on Ω.*

Proof. Let $\mu \in \Omega$. Then, as G acts transitively on Ω, there exists $g = kh \in KH = G$ such that $\omega^{kh} = \mu$. Since $k \in K = \mathrm{Stab}_G(\omega)$, we have that $\omega^k = \omega$. Hence $\omega^h = \mu$ and H acts transitively on Ω. $\square$

We require the following result in Chapter 3.

Lemma 1.1.18 *Let G be a finite group and $Q \trianglelefteq G$ be an elementary abelian 2-group. Suppose that $r \in G$ is an involution such that $C_Q(r) = [Q, r]$. Then:*

(i) every involution in rQ is conjugate to r; and

(ii) $|C_G(r)| = |C_Q(r)||C_{G/Q}(rQ)|$.

Proof. (i) Let $t \in rQ$ be an involution. Then $t = rq$, for some $q \in Q$. Since $t^2 = 1$, we have that $rqrq = 1$. As $r^{-1} = r$ and $q^{-1} = q$, this implies that $q^r = q$, and hence $q \in C_Q(r) = [Q, r]$. So $q = rq_1rq_1$, for some $q_1 \in Q$, and therefore $t = rrq_1rq_1 = r^{q_1}$ and t is conjugate to r.

(ii) Let $x \in C_G(r)$. Then $(rQ)^{xQ} = r^xQ = rQ$. So $xQ \in C_{G/Q}(rQ)$. This allows us to define a homomorphism, $\phi : C_G(r) \to C_{G/Q}(rQ)$ by $\phi(x) = xQ$. Clearly $\ker \phi =$

$C_Q(r)$. Next we show that ϕ is surjective. Let $yQ \in C_{G/Q}(rQ)$. Then $(rQ)^{yQ} = rQ$ and so $r^y \in rQ$. Thus r^y is an involution in rQ and, by (i), $r^y = r^q$ for some $q \in Q$. Therefore, $r^{yq^{-1}} = r^{yq} = r$. Hence, $yq \in \mathrm{C}_G(r)$. Since $yqQ = yQ$, we have that $\phi(yq) = yqQ = yQ$ and ϕ is surjective. Thus, by the First Isomorphism Theorem, $C_G(r)/C_Q(r) \cong C_{G/Q}(rQ)$ and $|C_G(r)| = |C_Q(r)||C_{G/Q}(rQ)|$, as required. □

We require the following definition in Chapters 3 and 6.

Definition 1.1.19 Suppose that $A \leq B \leq G$ are groups. Then:

(i) A is said to be weakly closed in B with respect to G if, whenever $A^g \leq B$ for $g \in G$, $A^g = A$; and

(ii) A is said to be strongly closed in B with respect to G if, for all $g \in G$, $A^g \cap B \leq A$.

We note that if A is strongly closed in B with respect to G then A is weakly closed in B with respect to G.

Lemma 1.1.20 *Let G be a finite group, p be a prime, $A \leq G$ be a p-subgroup and $B = N_G(A)$. Suppose that A is weakly closed in S with respect to G and $S \in \mathrm{Syl}_p(B)$ such that $A \leq S$. Then $N_G(S) \leq B$. In particular, $S \in \mathrm{Syl}_p(G)$.*

Proof. Let $S \in \mathrm{Syl}_p(B)$ and let $y \in N_G(S)$. We have that $A^y \leq S^y = S$. As A is weakly closed in S, $A^y = A$. Hence $y \in N_G(A) = B$. Therefore $N_G(S) \leq B$ and $S \in \mathrm{Syl}_p(G)$. □

The following theorem, which we require for the proof of Theorem 3.3.6, is due to Goldschmidt [12].

Theorem 1.1.21 *(Goldschmidt's Theorem) Let G be a finite group, $S \in \mathrm{Syl}_2(G)$ and A be an abelian subgroup of S such that A is strongly closed in S with respect to G. Suppose that $M = \langle A^G \rangle$. For $X \leq G$, define $\overline{X} = X/O_{2'}(M)$. Then:*

(i) $\overline{A} = O_2(\overline{M})\Omega_1(\overline{S})$*; and*

(ii) $\overline{M}$ *is a central product of an abelian* 2*-group and groups isomorphic to one of:* $\mathrm{L}_2(2^n)$ *for* $n \geq 2$*;* $\mathrm{U}_3(2^n)$ *for* $n \geq 2$*;* $\mathrm{Sz}(2^n)$ *for* $n > 1$ *and* n *odd;* $\mathrm{L}_2(q)$ *for* $q \cong 3, 5 \mod 8$*;* J_1*; or* ${}^2G_3(3^n)$ *for* $n > 1$ *and* n *odd.*

Proof. See [12]. □

Lemma 1.1.22 *Let* G *be a finite group,* $S \in \mathrm{Syl}_2(G)$ *and* A *be an abelian subgroup of* S *such that* A *is strongly closed in* S *with respect to* G*. Suppose that* $M = \langle A^G \rangle$ *and let* $\overline{M} = M/O_{2'}(M)$*. Then* $\overline{A}$ *is strongly closed in* $\overline{S}$ *with respect to* $\overline{G}$*.*

Proof. See [12, (2.12), page 78]. □

As mentioned in the introduction, we also require the following theorem of Parrott in order to prove the Corollary to Theorem B.

Theorem 1.1.23 *Let* G *be a finite group which contains an involution* t*. Suppose that* $G \neq C_G(t)O_{2'}(G)$ *and that* H *satisfies:*

(i) $R = O_2(C_G(t))$ *is extra-special of order* 2^9*; and*

(ii) $C_G(t)/R \cong \mathrm{Alt}(9)$*.*

Then $G \cong \mathrm{Th}$*.*

Proof. See [35]. □

We conclude this section with some results about non-central chief factors.

Definition 1.1.24 Let $H \leq G$. Suppose that there is a finite sequence of subgroups H_i, for $0 \leq i \leq n$ such that

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = G. \tag{1.1}$$

Then (1.1) is said to be a series of length n from H to G. The subgroups $H_0, H_1, \ldots, H_n$ are referred to as the terms of the sequence and the quotient groups H_i/H_{i-1} for $1 \leq i \leq n$ are called the factors of the series. Suppose that $H = 1$ and that $H_i \trianglelefteq G$ for all $0 \leq i \leq n$. Then (1.1) is said to be a normal series for G. Let

$$1 = J_0 \trianglelefteq J_1 \trianglelefteq \ldots \trianglelefteq J_m = G, \tag{1.2}$$

be a series of length m from H to G. Then (1.2) is said to be a refinement of (1.1) if (1.1) can be obtained by deleting terms of (1.2). A refinement is said to be a proper refinement if there exists $j \in \{0, 1, \ldots, m\}$ such that $H_i \neq J_j$ for $i = 0, 1, \ldots, n$.

A chief series of G is a minimal normal series of G with respect to refinement. The factors of a chief series are known as chief factors.

A chief factor H_i/H_{i-1} is said to be central if $[H_i/H_{i-1}, G] = 1$ and non-central otherwise.

Lemma 1.1.25 *Suppose that p is a prime, P is a p-group and G acts on P. Let $1 = P_0 \trianglelefteq P_1 \trianglelefteq \ldots \trianglelefteq P_{n-1} \trianglelefteq P_n = P$ be a G-invariant series of P. Set $\overline{P_i} = P_i/P_{i-1}$ for $i = 1, \ldots, n$. Then*

$$[P : C_P(G)] \geq \prod_{i=1}^{n} [\overline{P_i} : C_{\overline{P_i}}(G)].$$

Proof. See [31, Lemma 2.21]. □

Corollary 1.1.26 *Let p be a prime, P a p-group and G an operator group on P. Suppose that $[P : C_P(G)] = p^N$ for some natural number N. Then P has at most N non-central chief factors.*

Proof. This follows immediately from Lemma 1.1.25. □

1.2 Extra-special Groups

Definition 1.2.1 Suppose that P is a p-group for a prime p. Then P is said to be an extra-special group provided $|Z(P)| = |[P,P]| = |\Phi(P)| = p$.

Suppose that p is an odd prime and let $E = \langle x, y \mid x^p = y^p = 1, [x,y] \in Z(E)\rangle$ and $F = \langle x, y \mid x^{p^2} = y^p = 1, [x,y] = x^p\rangle$. If P is an extra-special group of order p^3 then:

(*i*) P is isomorphic to $\mathrm{Dih}(8)$ or Q_8 if $p = 2$; or

(*ii*) P is isomorphic to E or F if $p > 2$.

We note that $\mathrm{Q}_8 \circ \mathrm{Q}_8 \cong \mathrm{Dih}(8) \circ \mathrm{Dih}(8)$ and $F \circ F \cong E \circ F$, see [11, page 79]. Hence we can take central products of Q_8 and $\mathrm{Dih}(8)$ and E and F to get the following theorem.

Theorem 1.2.2 *Suppose that P is an extra-special group of order p^{1+2n}. Then exactly one of the following holds.*

(i) $p = 2$ and P is a central product of n copies of $\mathrm{Dih}(8)$. *We denote this group by 2_+^{1+2n}.*

(ii) $p = 2$ and P is a central product of $n-1$ copies of $\mathrm{Dih}(8)$ *and one copy of* Q_8. *We denote this group by 2_-^{1+2n}.*

(iii) $p \neq 2$, the exponent of P is p and P is a central product of n copies of E. We denote this group by p_+^{1+2n}.

(iv) $p \neq 2$, the exponent of P is p^2 and P is a central product of $n-1$ copies of E and one copy of F. We denote this group by p_-^{1+2n}.

Proof. See [11, Theorem 20.5]. □

We note that Theorem 1.2.2 implies that the group $\mathrm{Q}_8 \circ \mathrm{Q}_8 \circ \mathrm{Q}_8 \circ \mathrm{Q}_8 \cong 2_+^{1+8}$. This fact will be required in Chapter 6.

Lemma 1.2.3 *Suppose that Q is an extra-special group. If $X \leq Q$ such that $Z(Q) \leq X$, then $X \trianglelefteq Q$. In particular, if X, Y are subgroups of Q that contain $Z(Q)$, then X and Y normalize each other.*

Proof. We have that $[X, Q] \leq [Q, Q] = Z(Q) \leq X$ and so $X \trianglelefteq Q$. If X and Y are subgroups of X which contain $Z(Q)$, then X and Y are normal in Q. Hence the result follows. $\square$

We are particularly interested in the extra-special group 2_+^{1+4} and we now prove two results concerning this group.

Lemma 1.2.4 *Let $P \cong 2_+^{1+4}$. Then P contains exactly two subgroups that are isomorphic to* Q_8.

Proof. Since P can be written as the central product of two groups isomorphic to Q_8, there are at least two subgroups of P that are isomorphic to Q_8. We consider the elements of P of order 4. Let $P \cong Q_1 \circ Q_2$ where $Q_i \cong \mathrm{Q}_8$. This central product contains at least twelve elements of order 4, six in Q_1 and six in Q_2. Let $q_1q_2 \in P$ have order 4 with $q_1 \in Q_1$ and $q_2 \in Q_2$. Suppose that q_1 is central in Q_1. Then $q_1q_2 \in Q_2$. Similarly, if q_2 is central in Q_2 then $q_1q_2 \in Q_1$. So suppose that q_1 and q_2 both have order 4 and hence are not central elements. In other words, q_1q_2 is a new element of P of order 4. Then

$$\begin{aligned}
(q_1q_2)^2 &= q_1q_2q_1q_2 \\
&= q_1^2q_2^2 && \text{since } Q_1 \text{ and } Q_2 \text{ commute element-wise.} \\
&= 1 && \text{since } q_i^2 \text{ is central in } Q_i.
\end{aligned}$$

Hence q_1q_2 has order 2 which is a contradiction. Therefore q_1q_2 is either in Q_1 or Q_2 and P contains exactly twelve elements of order 4. Therefore P contains exactly two subgroups isomorphic to Q_8 and these commute element-wise. $\square$

In order to prove Lemma 1.2.7 we first require two results about non-degenerate symplectic forms.

Lemma 1.2.5 *Let Q be a non-abelian p-group for p a prime. Suppose that $\overline{Q} = Q/Z(Q)$ is elementary abelian and that $Q' = \langle z \rangle$ has order p. Define $f : \overline{Q} \times \overline{Q} \to \mathrm{GF}(p)$ by $f(\overline{x}, \overline{y}) = c$ where $[x, y] = z^c$ for $0 \leq c < p$. Then:*

(i) $(\overline{Q}, f)$ *is a non-degenerate symplectic* $\mathrm{GF}(p)$*-space;*

(ii) $|\overline{Q}| = p^{2n}$ *for some* $n \in \mathbb{N}$*;*

(iii) the largest possible order of an abelian subgroup of Q is $p^n.|Z(Q)|$; and

(iv) let $p = 2$ and $Z(Q)$ be elementary abelian. Define $q : \overline{Q} \to \mathrm{GF}(2)$ by $q(\overline{x}) = d$ where $x^2 = z^d$ for $0 \leq d < 2$. Then q is a quadratic form with respect to f.

Proof. See [31, Proposition 2.66]. □

We note that if Q is an extra-special group, then Q satisfies the hypotheses of Lemma 1.2.5.

Lemma 1.2.6 *Suppose V is a vector space over a field k and f is a non-degenerate symplectic form on V. If Q is a subgroup of the isometry group of (V, f), then $[V, Q]^{\perp} = C_V(Q)$.*

Proof. See [31, Lemma 2.53]. □

Lemma 1.2.7 *Let $P \cong 2^{1+4}_{+}$ and $\overline{P} = P/Z(P)$. Let x be an element of order 3 that acts non-trivially on $\overline{P}$. Then either:*

(i) $[C_P(x), [P, x]] = 1$ *and* $C_P(x) \cong [P, x] \cong \mathrm{Q}_8$*; or*

(ii) $C_P(x) \cong \langle t \rangle$*, where t is an involution.*

Proof. If $C_{\overline{P}}(x) = 1$, then $C_P(x) = \langle t \rangle$ and we are done. If $C_{\overline{P}}(x) = \overline{P}$, then x acts trivially on $\overline{P}$, giving us a contradiction. Therefore $C_P(x)$ and $[P, x]$ both have order at least 4.

Since x acts non-trivially on $\overline{P}$, we have that $\overline{P} = C_{\overline{P}}(x) \times [\overline{P}, x]$ by Coprime Action. Let $f : \overline{P} \times \overline{P} \to GF(2)$ be defined by $f(\overline{y}, \overline{z}) = c$ where $[y, z] = t^c$ for $0 \leq c < 2$. Then Lemma 1.2.5 implies that f is a non-degenerate symplectic form on $\overline{P}$. By Lemma 1.2.6, $C_{\overline{P}}(x)^{\perp} = [\overline{P}, x]$ and hence $C_{\overline{P}}(x)$ and $[\overline{P}, x]$ are non-degenerate with respect to the restriction of f and P. Therefore, there exists $\overline{a}, \overline{b} \in [\overline{P}, x]$ such that $f(\overline{a}, \overline{b}) \neq 0$. Thus $[a, b] \neq 0$. Hence $C_{\overline{P}}(x)$ and $[\overline{P}, x]$ are non-abelian.

Therefore both $C_P(x)$ and $[P, x]$ have order 8 and hence $[P, x] \cong \mathrm{Dih}(8)$ or Q_8. Since x acts non-trivially on $[P, x]$, we have that $[P, x] \ncong \mathrm{Dih}(8)$ and so $[P, x] \cong \mathrm{Q}_8$. We have that $[P, C_P(x), x] \leq [P', x] = [Z(P), x] = 1$ and $[C_P(x), x, P] = [1, P] = 1$. Therefore, by the Three Subgroup Lemma, $[x, P, C_P(x)] = [[P, x], C_P(x)] = 1$ and hence $C_P(x) \cong [P, x] \cong \mathrm{Q}_8$. □

We will need the following theorem in the case when $2n = 8$.

Theorem 1.2.8 *Suppose that* $P \cong 2_+^{1+2n}$. *Then* $\mathrm{Aut}(P)/\mathrm{Inn}(P) \cong \mathrm{GO}_{2n}^+(2)$ *and* $\mathrm{Aut}(P) \cong 2^{2n}.\mathrm{GO}_{2n}^+(2)$.

Proof. See [21]. □

1.3 Modules

We begin this section with two general results on modules.

Lemma 1.3.1 *Suppose that V is a* $\mathrm{GF}(p)$*-vector space and let x be an automorphism of V.*

(i) Then $V/C_V(x) \cong [V, x]$ *as* $\mathrm{GF}(p)$*-vector spaces.*

(ii) If $p = 2$ and x has order 2, then $C_V(x) \geq [V, x]$ and $|C_V(x)|^2 \geq |V|$.

Proof. (i) Suppose x is an automorphism of V and define, $\phi : V \to [V, x]$ by $\phi(v) = [v, x]$. Let $v, w \in V$ and $\lambda \in \mathrm{GF}(p)$. Then

$$\phi(\lambda v + w) = [\lambda v + w, x] = \lambda(v^x - v) + (w^x - w) = \lambda[v, x] + [w, x],$$

and hence ϕ is a linear transformation. Let $g \in [V, x]$. Then $g = \sum_i \lambda_i[v_i, x]$ for $\lambda_i \in \mathrm{GF}(p)$ and $v_i \in V$. We claim that ϕ is surjective. We have

$$\sum_i \lambda_i[v_i, x] = \sum_i [\lambda_i v_i, x] = \sum_i \phi(\lambda_i v_i) = \phi(\sum_i \lambda_i v_i),$$

since ϕ is a linear transformation. Hence ϕ is surjective.

We consider $\ker \phi$. Now,

$$\ker \phi = \{v \in V \mid [v, x] = 0\} = C_V(x).$$

Hence, the First Isomorphism Theorem implies that $V/C_V(x) \cong [V, x]$.

(ii) Suppose that $g \in [V, x]$. So $g = \sum_i \lambda_i[v_i, x]$ for $\lambda_i \in \mathrm{GF}(2)$ and $v_i \in V$. Hence, since ϕ is a linear transformation, $x^2 = 1$ and V is a $\mathrm{GF}(2)$-vector space, an easy calculation shows that

$$[g, x] = [\sum_i \lambda_i[v_i, x], x] = \sum_i (\lambda_i v_i)^{x^2} - 2(\lambda_i v_i)^x + \lambda_i v_i = 0.$$

Therefore $g \in C_V(x)$.

Suppose that $|V| = 2^a$, $|C_V(x)| = 2^b$ and $|[V, x]| = 2^c$. Then $2^{a-b} = 2^c$ and $2^b \geq 2^c$. So $2^{a-b} \leq 2^b$ and therefore $|C_V(x)|^2 = 2^{2b} \geq 2^a = |V|$ and the result holds. $\square$

Lemma 1.3.2 *Let W be a* $\mathrm{GF}(2)\langle x\rangle$*-module of even dimension where $\langle x\rangle$ is cyclic of order 2 and $U \subseteq W$ be $\langle x\rangle$-invariant of codimension 1. Assume* $\dim C_W(x) = \frac{1}{2}\dim W$. *Then $C_W(x) = C_U(x)$.*

Proof. Certainly $\dim C_U(x) \leq \dim C_W(x)$. So, by Lemma 1.3.1 (*ii*),

$$\dim C_W(x) \geq \dim C_U(x) \geq \lceil \frac{1}{2}\dim U \rceil = \frac{1}{2}\dim W = \dim C_W(x),$$

where $\lceil \frac{1}{2}\dim U \rceil$ denotes the smallest integer n such that $\frac{1}{2}\dim U \leq n$. Hence $C_U(x) = C_W(x)$. □

The following is seen in [31, Section 3.2] in which [4] is referred to.

Definition 1.3.3 Let $X \cong \mathrm{SL}_2(q)$ where $q = p^a$ for some prime p. Let R be the polynomial ring in two commuting indeterminates with coefficients in $\mathrm{GF}(q)$. Then R is a $\mathrm{GF}(q)X$-module. Define $R(i) = \{f \in R \mid f \text{ homogeneous of degree } i\}$. Then the $R(i)$ are submodules of R of dimension $i+1$. We see that $R(0)$ is the trivial $\mathrm{SL}_2(q)$-module and $R(1)$ is isomorphic to the natural $\mathrm{GF}(q)\mathrm{SL}_2(q)$-module. We also note that $R(2)$ is isomorphic to the adjoint $\mathrm{SL}_2(q)$-module which consists of 2×2 matrices of trace zero and is the 3-dimensional module which identifies $\mathrm{L}_2(q)$ with $\Omega_3(q)$. Therefore $R(0)$ and $R(1)$ are irreducible and $R(2)$ is irreducible if and only if q is odd.

We are particularly interested in natural $\mathrm{SL}_2(3)$ and natural $\Omega_3(3)$-modules. Natural $\mathrm{SL}_2(3)$-modules have dimension 2 and we see from Definition 1.3.3 that natural $\Omega_3(3)$-modules are irreducible and have dimension 3. The following result provides us with some useful facts about $\Omega_3(3)$-modules and Sylow 3-subgroups of $\Omega_3(3)$.

Lemma 1.3.4 *Let $G = \Omega_3(3)$. Suppose that V is a natural G-module and $S \in \mathrm{Syl}_3(G)$. Then $[V,S]$ has dimension 2, $[V,S,S]$ has dimension 1 and $C_V(S) = [V,S,S]$.*

Proof. We see from Definition 1.3.3 that we can represent V as 2×2 matrices that have trace zero. So suppose

$$V = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a, b, c \in \mathrm{GF}(3) \right\}.$$

This has dimension 3. We define a module structure on V by $v^g = g^{-1}vg$ for $v \in V$ and $g \in \mathrm{SL}_2(3)$. Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on V, we see that this becomes a module for $\Omega_3(3) \cong \mathrm{Alt}(4)$ under this module structure. Hence V is a 3-dimensional $\mathrm{GF}(3)G$-module. Let $S \in \mathrm{Syl}_3(G)$. So we may assume,

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \middle| \alpha \in \mathrm{GF}(3) \right\}.$$

Let $v \in V$ and $s \in S$. Then an easy calculation shows that,

$$[v, s] = v^s - v = \begin{pmatrix} \beta & 0 \\ \gamma & -\beta \end{pmatrix},$$

where $\beta = \alpha a$ and $\gamma = -\alpha^2 + \alpha a$. Since a and α we arbitrarily chosen, we see that β and γ are arbitrary elements of $\mathrm{GF}(3)$. Hence $[V, S] = \left\{ \begin{pmatrix} \beta & 0 \\ \gamma & -\beta \end{pmatrix} \middle| \beta, \gamma \in \mathrm{GF}(3) \right\}$ and therefore $[V, S]$ has dimension 2.

Now suppose that $w \in W = [V, S]$ and $s \in S$. Then again, an easy calculation shows that,

$$[w, s] = w^s - w = \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix},$$

where $\delta = -2\alpha\beta = \alpha\beta$. Hence $[W,S] = [V,S,S] = \left\{ \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} \middle| \delta \in \mathrm{GF}(3) \right\}$ and therefore $[V,S,S]$ has dimension 1.

Finally, $v \in C_V(S)$ if and only if $v^s = v$ for all $s \in S$. Let $s = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in S^{\#}$. So $\alpha \neq 0$ and we have

$$\begin{pmatrix} a + \alpha a & b \\ -\alpha^2 b + \alpha a + c & -\alpha a + a \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

Therefore $\alpha a = 0$. Since $\alpha \neq 0$ this implies that $a = 0$. Also $-\alpha^2 b + \alpha a + c = c$. Hence $b = 0$. Therefore $C_V(S) = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \middle| c \in \mathrm{GF}(3) \right\}$ and $C_V(S) = [V,S,S]$ as required.$\square$

Proposition 1.3.5 *Suppose that G is an elementary abelian 3-group of order 3^2 and that T is a Klein four-group acting faithfully on G. Then there exists $t \in T$ which inverts G and G contain two subgroups of order 3 that are invariant under the action of T and two that are not. In particular, we can consider G as a vector space of dimension 2 over $\mathrm{GF}(3)$ and T as the group generated by t_1 and t_2, where $t_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $t_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.*

Proof. By [13, Theorem 1.3.2], G is isomorphic to a vector space V of dimension 2 over $\mathrm{GF}(3)$. Hence V has basis $\{(1,0),(0,1)\}$. Therefore, we can consider T as a subgroup of $\mathrm{GL}_2(3)$ acting on V. Hence we may assume $T = \langle t_1, t_2 \rangle$ where $t_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $t_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $(a,b) \in V$ and consider the action of T on (a,b). We see that $(a,b)t \in V$ for $t \in T$ so V is invariant under the action of T. Since $(a,b)t_1t_2 = (-a,-b) = (a,b)^{-1}$ we see that V, and hence G is inverted by $t = t_1t_2 \in T$.

Now V contains four subspaces of dimension one, namely $\langle(1,0)\rangle$, $\langle(0,1)\rangle$, $\langle(1,1)\rangle$ and $\langle(1,-1)\rangle$. These correspond to the four subgroups of G of order 3. Suppose $(a,a) \in \langle(1,1)\rangle$. Then $(a,a)t_1 = (a,-a) \notin \langle(1,1)\rangle$ and clearly the subspace $\langle(1,1)\rangle$ is not T-invariant. Similarly, the subspace $\langle(1,-1)\rangle$ is not T-invariant. However $(a,0)t \in \langle(1,0)\rangle$ and $(0,b)t \in \langle(0,1)\rangle$ for all $t \in T$, $a,b \in \mathrm{GF}(3)$ and hence $\langle(1,0)\rangle$ and $\langle(0,1)\rangle$ are T-invariant subspaces of V. Hence, the subgroups of order 3 that correspond to these subspaces are the T-invariant subgroups of G. $\square$

1.4 Amalgams of Rank 2

Definition 1.4.1 An amalgam of rank 2 consists of three groups A_1, A_2 and B and two monomorphisms $\phi_i : B \to A_i$, for $i \in \{1,2\}$. We denote this amalgam by $\mathcal{A} = \mathcal{A}(A_1, A_2, B, \phi_1, \phi_2)$. If the groups A_1, A_2 and B are finite then $\mathcal{A}$ is said to be an amalgam of finite groups.

Definition 1.4.2 Let G be a group and $\mathcal{A} = \mathcal{A}(A_1, A_2, B, \phi_1, \phi_2)$ be an amalgam. Suppose that there exist monomorphisms $\psi_i : A_i \to G$, for $i \in \{1,2\}$ such that

$$\psi_1\phi_1 = \psi_2\phi_2 : B \to G.$$

Then G is said to be a host for $\mathcal{A}$. The group $H = \langle \operatorname{Im}\psi_1, \operatorname{Im}\psi_2\rangle$, is said to be a faithful completion of the amalgam $\mathcal{A}$. In particular, we note that a host for $\mathcal{A}$ contains a faithful completion of $\mathcal{A}$.

If G is a host for the amalgam $\mathcal{A}$, then we can identify the groups A_1, A_2 and B with their images in G. In this case $A_1 \cap A_2 \geq B$ and ϕ_i is regarded as the inclusion map of B into A_i. We denote this amalgam by $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$.

Definition 1.4.3 Let $\mathcal{A} = \mathcal{A}(A_1, A_2, B, \phi_1, \phi_2)$ be an amalgam of finite groups. We say that $\mathcal{A}$ is a simple amalgam if, whenever $K \leq B$, with $\phi_1(K) \trianglelefteq A_1$ and $\phi_2(K) \trianglelefteq A_2$, then $K = 1$.

1.5 The Coset Graph

Let $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$ be a simple amalgam and G be a faithful completion of $\mathcal{A}$. We identify A_1, A_2 and B with their images in G and regarding ϕ_i as the inclusion map of B into A_i. We also suppose that $B = A_1 \cap A_2$. In this section we construct the right coset graph of the amalgam $\mathcal{A}$ and prove various results concerning the action of subgroups of G on this graph.

Definition 1.5.1 The coset graph of the amalgam $\mathcal{A}$ is the graph $\Gamma = \Gamma(G, A_1, A_2, B)$ that has vertex set

$$V(\Gamma) = \{A_i g \mid g \in G, i \in \{1, 2\}\},$$

and edge set

$$E(\Gamma) = \{\{A_i g, A_j h\} \mid A_i g \cap A_j h \neq \emptyset, i \neq j\}.$$

The group G acts by right multiplication on $V(\Gamma)$ and $E(\Gamma)$ and hence acts on the graph Γ.

Notation 1.5.2 (*i*) For $\gamma \in V(\Gamma)$, let $G_\gamma = \mathrm{Stab}_G(\gamma)$.

(*ii*) For $\{\gamma, \delta\} \in \mathrm{E}(\Gamma)$, let $G_{\gamma\delta} = \mathrm{Stab}_G(\{\gamma, \delta\})$. So $G_{\gamma\delta} = G_\gamma \cap G_\delta$.

(*iii*) Let $d(\, , \,)$ be the distance metric on Γ.

(*iv*) For $\gamma \in V(\Gamma)$, let $\Gamma(\gamma) = \{\delta \in V(\Gamma) \mid \{\gamma, \delta\} \in E(\Gamma)\}$. In other words, $\Gamma(\gamma)$ is the set of vertices adjacent to the vertex γ in Γ.

Lemma 1.5.3 *(i) G acts faithfully on the graph Γ.*

(ii) G has two orbits on $V(\Gamma)$ and is transitive on $E(\Gamma)$.

(iii) For $\gamma \in V(\Gamma)$, G_γ is G-conjugate to either A_1 or A_2.

(iv) For $\{\gamma, \delta\} \in E(\Gamma)$, $G_{\gamma\delta}$ is G-conjugate to B.

Proof. ([31, Lemma 4.1])

(i) Suppose that K is the kernel of the action of G on $V(\Gamma)$. Then $K \leq G_\gamma$ for all $\gamma \in V(\Gamma)$. In particular, $K \lhd A_i$ for $i \in \{1, 2\}$. Hence $K = 1$ by the simplicity of $\mathcal{A}$ and therefore G acts faithfully on $V(\Gamma)$.

(ii) Let $\gamma = A_i g$ and $\delta = A_i h$. Since G acts by right multiplication on $V(\Gamma)$ we see that we can choose an $x \in G$ such that $A_i g x = A_i h$. Hence G acts transitively on $\{A_i g \mid g \in G\}$. Therefore G has two orbits on $V(\Gamma)$, namely the orbits of A_1 and A_2. If $\{\gamma, \delta\} \in E(\Gamma)$, then we may assume that $\gamma = A_i g$ and $\delta = A_j h$, where $\{i, j\} = \{1, 2\}$ and $g, h \in G$. So $A_i g \cap A_j h \neq \emptyset$ and hence, there exists $x \in A_i g \cap A_j h$. So $\{A_i g, A_j h\} = \{A_i x, A_j x\}$ and therefore $\{\gamma \cdot x, \delta \cdot x\}$ is an edge in Γ. Hence G acts transitively on $E(\Gamma)$.

(iii) Suppose that $\gamma = A_i g$, for $i \in \{1, 2\}$. So

$$G_\gamma = \{h \in G \mid A_i g h = A_i g\} = \{h \in G \mid g h g^{-1} \in A_i\} = \{h \in G \mid h \in A_i^g\} = A_i^g.$$

Hence G_γ is G-conjugate to A_i, for $i \in \{1, 2\}$.

(iv) Let $\{\gamma, \delta\} \in E(\Gamma)$. By (ii), we can choose $x \in A_i g \cap A_j h$. So $\{A_i g, A_j h\} = \{A_i x, A_j x\}$. So using (ii) we see that,

$$G_{\gamma\delta} = G_\gamma \cap G_\delta = A_i^g \cap A_j^h = A_i^x \cap A_j x = B^x,$$

and hence $G_{\gamma\delta}$ is G-conjugate to B. $\square$

Parts (*iii*) and (*iv*) of Lemma 1.5.3 imply that we can consider the amalgam $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$ as the amalgam $\mathcal{A}' = \mathcal{A}'(G_\gamma, G_\delta, G_{\gamma\delta})$ for $\{\gamma, \delta\} \in E(\Gamma)$.

Lemma 1.5.4 *(i) For $\gamma \in V(\Gamma)$, G_γ acts transitively on $\Gamma(\gamma)$. In particular, $|\Gamma(\gamma)| = |G_\gamma : G_{\gamma\delta}|$ for any $\delta \in \Gamma(\gamma)$.*

(ii) The graph Γ is connected.

Proof. ([31, Lemmas 4.3 and 4.5])

(i) We may assume that $\gamma = A_i$, for $i \in \{1, 2\}$. Let $\delta, \tau \in \Gamma(\gamma)$. Hence $\delta = A_j g_1$ and $\tau = A_j g_2$, where $\{i, j\} = \{1, 2\}$ and $g_1, g_2 \in G$. Also, $A_i \cap A_j g_1 \neq \emptyset \neq A_i \cap A_j g_2$. So we have that $A_j g_1 = A_j x_1$ and $A_j g_2 = A_j x_2$, for some $x_1, x_2 \in A_i$. Therefore, $\delta \cdot g = \tau$ for $g = x_1^{-1} x_2 \in A_i$. So A_i acts transitively on $\Gamma(\gamma)$. By Lemma 1.5.3, G_γ is conjugate to A_i.

By the orbit-stabilizer theorem we have that, $|G_\gamma| = |\Gamma(\gamma)||G_{\gamma\delta}|$. So $|\Gamma(\gamma)| = |G_\gamma|/|G_{\gamma\delta}| = |G_\gamma : G_{\gamma\delta}|$.

(ii) Let Φ be the connected component of Γ that contains the edge $\{\gamma, \delta\}$, where $\gamma = A_1$ and $\delta = A_2$. Then $\langle G_\gamma, G_\delta \rangle = \langle A_1, A_2 \rangle = G$ stabilizes Φ. Let $\tau \in V(\Gamma)$. Then $\tau = A_i g$ for $i \in \{1, 2\}$ and $g \in G$. Hence $\tau \in \{\gamma \cdot g, \delta \cdot g\}$. So $\gamma \in \Phi$ and hence $\Gamma = \Phi$ and in particular, Γ is connected. $\square$

We now introduce some more subgroups of G.

Notation 1.5.5 Suppose that $\{\gamma, \delta\} \in E(\Gamma)$ and that p is a prime. Then we define $L_\gamma = O^{p'}(G_\gamma)$, $Q_\gamma = O_p(L_\gamma)$, $Z_\gamma = \Omega_1(Z(Q_\gamma))$ and $S_{\gamma\delta} = O_p(G_{\gamma\delta})$.

Definition 1.5.6 We define $b = \min\limits_{\gamma,\delta \in V(\Gamma)} \{d(\gamma, \delta) \mid Z_\gamma \nleq Q_\delta\}$. We call b the critical distance of Γ. Any pair of vertices $\{\gamma, \delta\}$ is called a critical pair if $d(\gamma, \delta) = b$ and $Z_\gamma \nleq Q_\delta$.

1.6 Weak BN-pairs of Characteristic p

Definition 1.6.1 [9, page 94, Hypothesis A] Let $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$ be a simple amalgam of finite groups and p be a prime. Suppose that there exists a normal subgroup, A_i^* of A_i, for $i \in \{1, 2\}$, such that:

(*i*) $O_p(A_i) \leq A_i^*$ and $A_i = A_i^* B$;

(*ii*) A_i is p-constrained; and

(*iii*) $A_i^* \cap B$ is the normalizer of a Sylow p-subgroup of A_i^* and, for $n_i \geq 1$, $A_i^*/O_p(A_i)$ is isomorphic to one of:

(a) $\mathrm{L}_2(p^{n_i})$, $\mathrm{SL}_2(p^{n_i})$, $\mathrm{U}_3(p^{n_i})$ or $\mathrm{SU}_3(p^{n_i})$, for $p \geq 2$;

(b) ${}^2\mathrm{B}_2(2^{n_i})$ or $\mathrm{Dih}(10)$, for $p = 2$; or

(c) ${}^2\mathrm{G}_2(3^{n_i})$ or ${}^2\mathrm{G}_2(3)'$, for $p = 3$.

Then we say that the amalgam $\mathcal{A}$ is a rank 2, weak BN-pair of characteristic p with respect to A_1, A_2 and B.

We say that an amalgam $\mathcal{A}$ is a weak BN-pair of type G, where G is a group, if it originates from the group G. In other words, if G is a faithful completion of the amalgam $\mathcal{A}$. We note that G is not the only completion of an amalgam of type G.

We note that a complete list of types of weak BN-pairs of characteristic p can be found in [9].

Throughout this work we will be interested in amalgams of types $\mathrm{G}_2(3)$ and F_3, the full definitions of which are given below. The uniqueness of these types of amalgams follow from [9] and [10] respectively. We note that these are weak BN-pairs of characteristic 3.

Definition 1.6.2 [32, Definition 2.1] Let $\mathcal{A} = \mathcal{A}(G_\alpha, G_\beta, G_{\alpha\beta})$ be a simple amalgam of finite groups, $L_\gamma = O^{3'}(G_\gamma)$ for $\gamma \in \{\alpha, \beta\}$ and $L_{\alpha\beta} = (L_\alpha \cap G_{\alpha\beta}) \cap (L_\beta \cap G_{\alpha\beta})$. Suppose that $\mathrm{Syl}_3(L_{\alpha\beta}) \subseteq \mathrm{Syl}_3(L_\alpha) \cap \mathrm{Syl}_3(L_\beta)$. Then $\mathcal{A}$ is of type $\mathrm{G}_2(3)$ if the following hold for $\gamma \in \{\alpha, \beta\}$.

(*i*) For $\{\gamma, \delta\} = \{\alpha, \beta\}$, $G_\gamma = (L_\delta \cap G_{\alpha\beta})L_\gamma$.

(*ii*) $L_\gamma/Q_\gamma \cong \mathrm{SL}_2(3)$.

(*iii*) Q_γ has order 3^5.

(*iv*) Z_γ has order 3^3.

(*v*) $Q'_\gamma = Z(L_\gamma)$.

(*vi*) As L_γ/Q_γ-modules, $Z_\gamma/Z(L_\gamma)$ and Q_γ/Z_γ are natural $\mathrm{SL}_2(3)$-modules.

Definition 1.6.3 [32, Definition 2.1] Let $\mathcal{A} = \mathcal{A}(G_\alpha, G_\beta, G_{\alpha\beta})$ be a simple amalgam of finite groups, $L_\gamma = O^{3'}(G_\gamma)$ for $\gamma \in \{\alpha, \beta\}$ and $L_{\alpha\beta} = (L_\alpha \cap G_{\alpha\beta}) \cap (L_\beta \cap G_{\alpha\beta})$. Suppose that $\mathrm{Syl}_3(L_{\alpha\beta}) \subseteq \mathrm{Syl}_3(L_\alpha) \cap \mathrm{Syl}_3(L_\beta)$. Then $\mathcal{A}$ is of type F_3 if the following hold.

(*i*) For $\{\gamma, \delta\} = \{\alpha, \beta\}$, $G_\gamma = (L_\delta \cap G_{\alpha\beta})L_\gamma$.

(*ii*) For $\gamma \in \{\alpha, \beta\}$, $L_\gamma/Q_\gamma \cong \mathrm{SL}_2(3)$.

(*iii*) There exist normal subgroups of L_α,

$$1 < Z_\alpha < U_\alpha < Q_\alpha = O_3(L_\alpha),$$

such that, as L_α/Q_α-modules:

(a) Z_α is a natural $\mathrm{SL}_2(3)$-module;

(b) U_α/Z_α is an $\Omega_3(3)$-module of order 3^3; and

(c) Q_α/U_α is indecomposable and has two composition factors, each of which is a natural $\mathrm{SL}_2(3)$-module.

(iv) There exist normal subgroups of L_β,

$$1 < Z_\beta < V_\beta < Z(W_\beta) < W_\beta < C_\beta = C_{L_\beta}(V_\beta) < Q_\beta = O_3(L_\beta),$$

such that, as L_β/Q_β-modules:

(a) Z_β, $Z(W_\beta)/V_\beta$ and C_β/W_β all have order 3 and are centralized by L_β; and

(b) V_β/Z_β, $W_\beta/Z(W_\beta)$ and Q_β/C_β are all natural $\mathrm{SL}_2(3)$-modules.

(v) Let $X \to^\gamma Y$ mean $\langle X^{L_\gamma}\rangle = Y$. Then

$$Z_\beta \to^\alpha Z_\alpha \to^\beta V_\beta \to^\alpha U_\alpha \to^\beta W_\beta \to^\alpha Q_\alpha.$$

If a completion G of an amalgam of type $\mathrm{G}_2(3)$ or F_3 is defined then we can re-define $L_{\alpha\beta}$ as the intersection of L_α and L_β. We note that by [9] and [10] we see that an amalgam of type $\mathrm{G}_2(3)$ or F_3 is unique up to isomorphism, although we will not use this fact and require only Definitions 1.6.2 and 1.6.3.

The structure of the subgroups of the amalgams defined in Definitions 1.6.2 and 1.6.3 are depicted in Figures 1.1 and 1.2 respectively.

1.7 p-generated Amalgams

In this section we prove two important results, namely Lemma 1.7.2 and Theorem 1.7.3, that will be required in Chapter 6.

Definition 1.7.1 A simple amalgam $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$ is said to be p-generated for some prime p provided:

(I) $A_i = (O^{p'}(A_j) \cap B)O^{p'}(A_i)$, for $i \neq j$;

(II) $O^{p'}(A_i) = \langle X^{A_i}\rangle$, for any p-subgroup X of B with $X \not\leq O_p(A_i)$; and

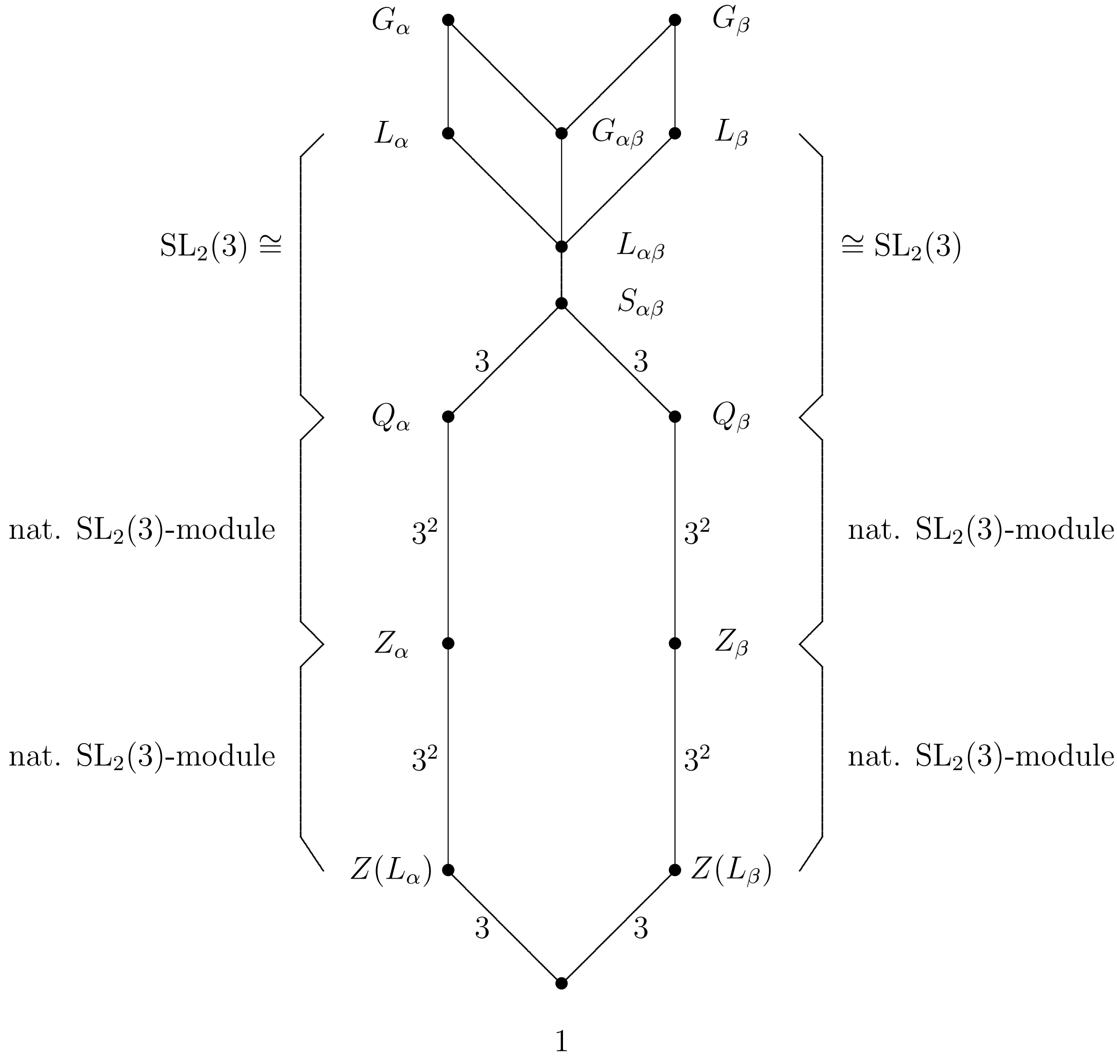

Figure 1.1: Partial Subgroup Lattice–Amalgams of Type $\mathrm{G}_2(3)$.

(III) $O_p(A_i) \nleq O_p(A_j)$, for $i \neq j$.

Lemma 1.7.2 *Amalgams of type* $\mathrm{G}_2(3)$ *are* 3*-generated.*

Proof. Let $\mathcal{A} = \mathcal{A}(G_\alpha, G_\beta, G_{\alpha\beta})$ be an amalgam of type $\mathrm{G}_2(3)$. By Definition 1.6.2 part (i),

$$G_\gamma = (L_\delta \cap G_{\alpha\beta})L_\gamma = (O^{3'}(G_\alpha) \cap G_{\alpha\beta})O^{3'}(G_\gamma),$$

for $\{\gamma, \delta\} = \{\alpha, \beta\}$. Hence Definition 1.7.1(I) holds.

Now let $\gamma \in \{\alpha, \beta\}$ and X be a 3-subgroup of $G_{\alpha\beta}$ such that $X \nleq O_3(G_\gamma)$. Suppose

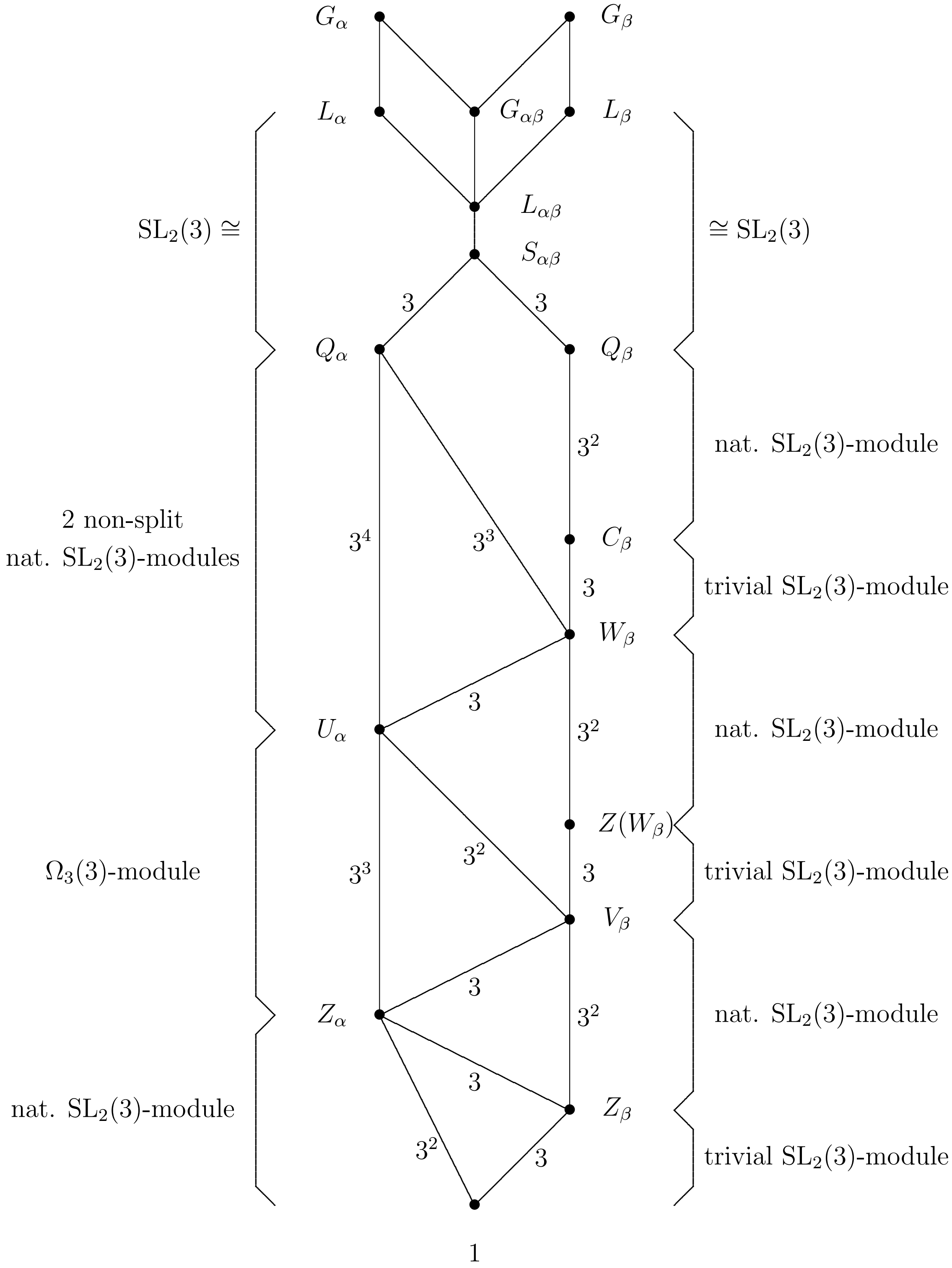

Figure 1.2: Partial Subgroup Lattice–Amalgams of Type F_3.

that $G_\gamma/\langle X^{G_\gamma}\rangle$ is not a $3'$-group. Since $S_{\alpha\beta} = XQ_\gamma$,

$$Q_\gamma\langle X^{G_\gamma}\rangle/\langle X^{G_\gamma}\rangle = S_{\alpha\beta}\langle X^{G_\gamma}\rangle/\langle X^{G_\gamma}\rangle \in \mathrm{Syl}_3(G_\gamma/\langle X^{G_\gamma}\rangle).$$

Also, 3 divides $|G_\gamma/\langle X^{G_\gamma}\rangle|$ as it is not a $3'$-group and hence,

$$Q_\gamma\langle X^{G_\gamma}\rangle/\langle X^{G_\gamma}\rangle \cong Q_\gamma/(Q_\gamma \cap \langle X^{G_\gamma}\rangle) \neq 1.$$

Since $[Q_\gamma, \langle X^{G_\gamma}\rangle] \leq Q_\gamma \cap \langle X^{G_\gamma}\rangle$ we have that $Q_\gamma/(Q_\gamma \cap \langle X^{G_\gamma}\rangle)$ has only central chief factors for $\langle X^{G_\gamma}\rangle$. ¿From Definition 1.6.2 (vi) we see that $|Q_\gamma/Z(L_\gamma)| = 3^4$ and $Q_\gamma/Z(L_\gamma)$ contains two non-central chief factors. Hence $|Q_\gamma/(Q_\gamma \cap \langle X^{G_\gamma}\rangle)| \leq 3$ and, in particular, as $|Z(L_\gamma)| = 3$, we have that $Z(L_\gamma) \nleq Q_\gamma \cap \langle X^{G_\gamma}\rangle$. So

$$[Q_\gamma \cap \langle X^\gamma\rangle, Q_\gamma \cap \langle X^\gamma\rangle] \leq [Q_\gamma, Q_\gamma] \cap (Q_\gamma \cap \langle X^{G_\gamma}\rangle) = Z(L_\gamma) \cap (Q_\gamma \cap \langle X^{G_\gamma}\rangle) = 1,$$

and hence $Q_\gamma \cap \langle X^\gamma\rangle$ is abelian. Thus $Q_\gamma = Z(L_\gamma)(Q_\gamma \cap \langle X^{G_\gamma}\rangle)$ is also abelian, which contradicts $Q_\gamma' = Z(L_\gamma)$ in Definition 1.6.2 (v). Therefore $G_\gamma/\langle X^{G_\gamma}\rangle$ is a $3'$-group. Since $X \leq O^{3'}(G_\gamma)$ we infer that $O^{3'}(G_\gamma) = \langle X^{G_\gamma}\rangle$. So Definition 1.7.1 (II) holds.

If $Q_\alpha = Q_\beta$, then $Q_\gamma \trianglelefteq \langle G_\alpha, G_\beta\rangle$ for $\gamma \in \{\alpha, \beta\}$, contradicting the simplicity of $\mathcal{A}$. So Definition 1.7.1 (III) holds. Hence all conditions hold and $\mathcal{A}$ is a 3-generated amalgam.□

Theorem 1.7.3 *Let $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$ be a p-generated amalgam and H be a host for $\mathcal{A}$. Suppose that H satisfies:*

(i) $O_{p'}(H) = 1$;

(ii) $\mathrm{Syl}_p(H) \supseteq \mathrm{Syl}_p(B)$; *and*

(iii) For $S \in \mathrm{Syl}_p(B)$, $N_H(S) \leq \langle A_1, A_2\rangle$.

Then H is a non-abelian simple group.

Proof. We note that if H is a host for $\mathcal{A}$, this implies that $H \geq \langle A_1, A_2 \rangle$.

Let K be a minimal normal subgroup of H. Suppose that p does not divide $|K|$. Then $K \leq O_{p'}(H) = 1$, which contradicts $K \neq 1$. Now suppose that K has order p^n for some n. Then by (ii), $K \leq B$. Therefore, K is normal in both A_1 and A_2, contradicting the simplicity of $\mathcal{A}$. Hence K is neither a p-group, nor a p'-group.

Let $S \in \mathrm{Syl}_p(B)$. Then $S \cap K \in \mathrm{Syl}_p(K)$. Since K is not a p'-group we have that $S \cap K \neq 1$. Suppose that $S \cap K \leq O_p(A_1) \cap O_p(A_2)$. Then as K is normal in H,

$$1 \nleq S \cap K = O_p(A_1) \cap K = O_p(A_2) \cap K$$

is normalized by both A_1 and A_2. This contradicts the simplicity of $\mathcal{A}$.

So, without loss of generality, we may assume that $S \cap K \nleq O_p(A_1)$. Therefore, by Definition 1.7.1 (II),

$$O^{p'}(A_1) \leq \langle (S \cap K)^{A_1} \rangle \leq \langle K^{A_1} \rangle = K,$$

and thus $O_p(A_1) = O_p(O^{p'}(A_1)) \leq S \cap K$. So Definition 1.7.1 (III) gives us $S \cap K \nleq O_p(A_2)$. Hence $O^{p'}(A_2) \leq K$ and therefore $\langle A_1, A_2 \rangle \leq K$ by Definition 1.7.1 (I).

By the Frattini argument, $H = KN_H(S)$, By (iii),

$$N_H(S) \leq \langle A_1, A_2 \rangle \leq K.$$

Hence $H = K$ and in particular, H is a minimal normal subgroup of itself. Therefore, H is a simple group. Since K is neither a p-group, nor a p'-group, K, and hence H, cannot have prime power order. Therefore H is non-abelian. ☐

Chapter 2

A Recognition Result for $\mathrm{G}_2(3)$

In this chapter we prove a result which enables us to recognise $\mathrm{G}_2(3)$ from the structure of its Sylow 3-subgroups. We require this result to prove Theorem A in Chapter 5. We note that this is the only section of this work which requires a $\mathcal{K}$-group hypothesis. First we require the following hypothesis.

Hypothesis 2.0.1 Let G be a non-abelian finite simple $\mathcal{K}$-group and $S \in \mathrm{Syl}_3(G)$. Suppose that:

(*i*) $|S| = 3^6$;

(*ii*) $|Z(S)| = 3^2$;

(*iii*) S has exponent 9;

(*iv*) the maximal order of an elementary abelian 3-subgroup of G is 3^4; and

(*v*) there is more than one elementary abelian 3-subgroup of G of order 3^4.

The following is the main result of this chapter.

Theorem 2.0.2 *Suppose that G satisfies the conditions in Hypothesis 2.0.1. Then*

$$G \cong \mathrm{G}_2(3).$$

In order to prove Theorem 2.0.2, we assume that a group G satisfies the conditions in Hypothesis 2.0.1, and show that the only possibility is $G \cong \mathrm{G}_2(3)$. Clearly $\mathrm{G}_2(3)$ contains a Sylow 3-subgroup which satifies the conditions in Hypothesis 2.0.1. We use the $\mathcal{K}$-group hypothesis to see that we have the following three cases to consider for G:

1. $G \cong \mathrm{Alt}(n)$ for some $n \geq 5$;

2. G is isomorphic to a Lie type group over a field k; or

3. G is isomorphic to a sporadic simple group.

We now look at each case in turn.

2.1 G Isomorphic to $\mathrm{Alt}(n)$

Table 2.1 shows the orders of the Sylow 3-subgroups, S of $\mathrm{Alt}(n)$, for $n \leq 18$.

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\lvert S\rvert$	3	3^2	3^2	3^2	3^4	3^4	3^4	3^5	3^5	3^5	3^6	3^6	3^6	3^8

Table 2.1: Order of Sylow 3 Subgroups of $\mathrm{Alt}(n)$

Lemma 2.1.1 *Suppose G satisfies the conditions in Hypothesis 2.0.1. Then $G \not\cong \mathrm{Alt}(n)$.*

Proof. Since $\mathrm{Alt}(n) \leq \mathrm{Alt}(n+1)$ we have that $|\mathrm{Alt}(n)|_3 \leq |\mathrm{Alt}(n+1)|_3$. Hence, from Table 2.1, we see that we only need to consider the Sylow 3-subgroups of $\mathrm{Alt}(15)$. Suppose $S \in \mathrm{Syl}_3(\mathrm{Alt}(15))$ and let

$$H \cong \mathrm{Alt}(6) \times \mathrm{Alt}(9) \leq \mathrm{Alt}(15) \cong G.$$

So, $|H|_3 = |G|_3$, we see that H contains a Sylow 3-subgroup of $\mathrm{Alt}(15)$. A Sylow 3-subgroup, T of $\mathrm{Alt}(6)$ has order 3^2 and is abelian. Hence $T \leq Z(S)$. Suppose $U \in \mathrm{Syl}_3(\mathrm{Alt}(9))$. Then $Z(U) \neq 1$. Hence $Z(T) \times Z(U) \leq Z(S)$. However $|Z(T) \times Z(U)| \geq 3^3$

and therefore $|Z(S)| \geq 3^3$, which is a contradiction. Hence $G \not\cong \mathrm{Alt}(15)$ and consequently $G \not\cong \mathrm{Alt}(n)$. $\square$

2.2 G Isomorphic to a Group of Lie Type

This section deals with the groups of Lie type, first in characteristic 3 and then in characteristic not equal to 3. Throughout these sections ${}^mG(r^a)$ denotes a group of Lie type over the field of order r^a, where $m \in \{1,2,3\}$ denotes any twisting of the group. We also suppose $U \in \mathrm{Syl}_r({}^mG(r^a))$. We let Σ be the root system associated with ${}^mG(r^a)$ with Σ^+ denoting the set of positive roots of Σ with respect to a set of fundamental roots Π. For more details see [7].

2.2.1 Characteristic of k is 3

We start with some preliminary lemmas.

Proposition 2.2.1 *Suppose that ${}^mG(r^a)$ is a group of Lie type and that $U \in \mathrm{Syl}_r({}^mG(r^a))$. Then either:*

(i) $|U| = (r^a)^{|\Sigma^+|}$; *or*

(ii) ${}^mG(r^a)$ *is* ${}^2\mathrm{G}_2(3^a)$, ${}^2\mathrm{F}_4(2^a)$ *or* ${}^2\mathrm{B}_2(2^a)$.

Proof. See [7, Theorem 5.3.3, (ii), Lemma 14.1.2]. $\square$

Since in this section we are concerned with fields of characteristic 3, we consider ${}^2\mathrm{F}_4(2^a)$ or ${}^2\mathrm{B}_2(2^a)$ in Section 2.2.2. Also, since ${}^2\mathrm{G}_2(3)$ is not simple and $|{}^2\mathrm{G}_2(3^a)|_3 \geq 3^9$ for $a \geq 3$, we can eliminate these cases straight away. Hence we only need to consider case (i) of Proposition 2.2.1. So, in order to find groups of Lie type that have Sylow 3-subgroups of order 3^6 we just need to know the number of positive roots of the underlying Lie algebra. These can be seen in a table in [7, Section 3.6]. So, using Proposition 2.2.1, we see that G

is isomorphic to one of: $\mathrm{A}_3(3) \cong \mathrm{L}_4(3)$; $\mathrm{A}_2(3^2) \cong \mathrm{L}_3(3^2)$; $\mathrm{A}_1(3^6) \cong \mathrm{L}_2(3^6)$; ${}^2\mathrm{A}_3(3) \cong \mathrm{U}_4(3)$; ${}^2\mathrm{A}_2(3^2) \cong \mathrm{U}_3(3^2)$; or $\mathrm{G}_2(3)$.

In order to eliminate some of these cases we require an additional proposition.

Proposition 2.2.2 *Suppose that ${}^mG(r^a)$ is a group of Lie type but not ${}^2\mathrm{G}_2(3^a)$, ${}^2\mathrm{F}_4(2^a)$ or ${}^2\mathrm{B}_2(2^a)$. Let $U \in \mathrm{Syl}_r({}^mG(r^a))$. Then either $|Z(U)| = r^a$ or ${}^mG(r^a)$ is $\mathrm{F}_4(2^a)$, $\mathrm{C}_n(2^a)$ or $\mathrm{G}_2(3^a)$ and $|Z(U)| = r^{2a}$.*

Proof. See [17, Theorem 3.3.1]. □

Lemma 2.2.3 *Suppose that G satisfies the conditions in Hypothesis 2.0.1. Then $G \not\cong \mathrm{L}_4(3)$ or $\mathrm{U}_4(3)$.*

Proof. Suppose that G is isomorphic to $\mathrm{L}_4(3) \cong \mathrm{A}_3(3)$ or $\mathrm{U}_4(3) \cong {}^2\mathrm{A}_3(3)$. Then by Proposition 2.2.2, if $S \in \mathrm{Syl}_3(G)$, then $|Z(S)| = 3$. This contradicts the hypothesis that $|Z(S)| = 3^2$. Hence $G \not\cong \mathrm{L}_4(3)$ or $\mathrm{U}_4(3)$. □

Lemma 2.2.4 *Suppose that G satisfies the conditions in Hypothesis 2.0.1. Then $G \not\cong \mathrm{L}_3(3^2)$ or $\mathrm{U}_3(3^2)$.*

Proof. Suppose that G is isomorphic to $\mathrm{L}_3(3^2) \cong \mathrm{A}_2(3^2)$ or $\mathrm{U}_3(3^2) \cong {}^2\mathrm{A}_2(3^2)$ and $S \in \mathrm{Syl}_3(G)$. By Proposition 2.2.2, $|Z(S)| = 3^2$. Since we can represent $\mathrm{L}_3(3^2)$ and $\mathrm{U}_3(3^2)$ by 3×3 matrices over a field of characteristic 3, we see that the elements of S are lower triangular matrices. So

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 3a & 1 & 0 \\ 3b+3ac & 3c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b, c are elements of the field of order 3^2. Hence all elements in S have order 3. This contradicts the fact that S contains elements of order 9 and hence $G \not\cong \mathrm{L}_3(3^2)$ or $\mathrm{U}_3(3^2)$. $\square$

Lemma 2.2.5 *Suppose that G satisfies the conditions in Hypothesis 2.0.1. Then $G \not\cong \mathrm{L}_2(3^6)$.*

Proof. Suppose that $G \cong \mathrm{L}_2(3^6) \cong \mathrm{A}_1(3^6)$. Let $S \in \mathrm{Syl}_3(G)$. Then by Proposition 2.2.2, $|Z(S)| = 3^6$. This contradicts the hypothesis that $|Z(S)| = 3^2$ and hence $G \not\cong \mathrm{L}_2(3^6)$. $\square$

2.2.2 Characteristic of k is not 3

Let ${}^mG(r^a)$ be a group of Lie type over the field of order r^a. Then provided ${}^mG(r^a)$ is not ${}^2\mathrm{F}_4(2^a)$ or ${}^2\mathrm{B}_2(2^a)$,

$$|{}^mG(r^a)| = |U| \prod_i \Phi_i(r^a)^{n_i}, \tag{2.1}$$

where $\Phi_i(r^a)$ is the cyclotomic polynomial for the i^{th} root of unity, the n_i are non-negative integers, almost all zero and U is a Sylow r-subgroup of ${}^mG(r^a)$, [17, Section 4.10].

Definition 2.2.6 We define m_0 to be the multiplicative order of r^a modulo p, where $p \neq r$. Therefore n_{m_0} is a non-negative integer such that $\Phi_i(r^a)$ occurs to the power of n_{m_0} in Equation 2.1.

Lemma 2.2.7 *Let ${}^mG(r^a)$ be a group of Lie type and $p \neq r$ be a prime. Then:*

(i) the p-rank of ${}^mG(r^a)$ is $m_p({}^mG(r^a)) = n_{m_0}$ or $n_{m_0} - 1$; and

(ii) a Sylow p-subgroup of ${}^mG(r^a)$ has a unique elementary abelian subgroup of rank $m_p({}^mG(r^a))$, unless $p = 3$ and ${}^mG(r^a)$ is isomorphic to one of: $\mathrm{A}_2(r^a)$; ${}^2\mathrm{A}_2(r^a)$; $\mathrm{G}_2(r^a)$; ${}^2\mathrm{F}_4(r^a)$; or ${}^3\mathrm{D}_4(r^a)$.

Proof. See [17, Theorem 4.10.3, parts (b) and (c)]. $\square$

Since by Hypothesis 2.0.1, G has more that one maximal elementary abelian 3-subgroup of maximal rank we see that we just need to consider the five exceptions in Lemma 2.2.7.

Since $p = 3$ we have that $m_0 = 1$ when $r^a \equiv 1 \mod 3$ or $m_0 = 2$ when $r^a \equiv 2 \mod 3$. By Hypothesis 2.0.1, $n_{m_0} \in \{4, 5\}$ and so we are looking for a fourth or fifth power of $\Phi_1(r^a)$ or $\Phi_2(r^a)$ occurring in the factorization in Equation 2.1 for each of the five cases in case (ii) of Lemma 2.2.7 to be a possibility for G. We show that these powers cannot occur.

Lemma 2.2.8 *Suppose that G satisfies the conditions in Hypothesis 2.0.1. Then G is not isomorphic to any of the exceptions listed in Lemma 2.2.7, (ii) with $r \neq 3$.*

Proof. Let $q = r^a$. By [14, Tables 10:1 and 10:2] we have that:

(i) $|\mathrm{A}_2(q)| = q^2\Phi_1(q)^2\Phi_2(q)\Phi_3(q)$;

(ii) $|{}^2\mathrm{A}_2(q)| = q^2\Phi_1(q)\Phi_2(q)^2$;

(iii) $|\mathrm{G}_2(q)| = q^6\Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)\Phi_6(q)$;

(iv) $|{}^2\mathrm{F}_4(q)| = q^{24}\Phi_1(q)^2\Phi_2(q)^2\Phi_4(q)^2\Phi_6(q)\Phi_{12}(q)$; and

(v) $|{}^3\mathrm{D}_4(q)| = q^{12}\Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)^2\Phi_6(q)^2\Phi_{12}(q)$.

We see that none of these give us $n_{m_0} \in \{4, 5\}$ and hence G cannot be isomorphic to any of the exceptions listed in Lemma 2.2.7. $\square$

2.3 G Isomorphic to a Sporadic Simple Group

By considering the orders of the sporadic simple groups, the possibilities in this case are:

(i) $G \cong \mathrm{HN}$;

(*ii*) $G \cong \mathrm{M^cL}$;

(*iii*) $G \cong \mathrm{Co}_2$.

These are chosen by simply considering the orders of a Sylow 3-subgroup.

We look at these possibilities in turn and show that none of them can occur.

Lemma 2.3.1 *Suppose that G satisfies the conditions in Hypothesis 2.0.1. Then $G \ncong$ HN.*

Proof. Suppose that $G \cong \mathrm{HN}$ and let $X = 3_+^{1+4} : 4.\mathrm{Alt}(5)$, one of the maximal subgroups of HN, [8, page 166]. Let $Q = O_3(X) \cong 3_+^{1+4}$. So $C_X(Q) \trianglelefteq X$. Also let $S \in \mathrm{Syl}_3(\mathrm{HN})$, so $|S| = 3^6$. Now suppose that $|Z(S)| \geq 3^2$. So $Z(S) \nleq Q$ since $|Z(Q)| = 3$ as Q is an extra-special group. Note that $Z(S) \leq C_X(Q)$. This implies that $C_X(Q) \nleq Q$ and $C_X(Q)Q/Q \trianglelefteq 4.\mathrm{Alt}(5)$. Now 3 divides $|4.\mathrm{Alt}(5)|$ and $|Z(S)Q/Q| = 3$. Hence $C_X(Q)$ has a composition factor that is isomorphic to $\mathrm{Alt}(5)$. In particular $C_X(Q)$ has even order and so $C_X(Q)$ contains an involution, i. There are two conjugacy classes of involutions in G and $|C_G(i)|$ is equal to either $177408000 = 2^{11} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ or $3686400 = 2^{14} \cdot 3^2 \cdot 5$, see [8, page 164]. Now $|Q| = 3^5$ and $Q \leq C_X(i)$. However the orders of the centralizers of the classes of involutions are not divisible by 3^5. Hence we have a contradiction to $|Z(S)| \geq 3^2$ and so $|Z(S)| = 3$. Thus $G \ncong \mathrm{HN}$. □

The following corollary follows from Lemma 2.2.3.

Corollary 2.3.2 *Suppose that G satisfies the conditions in Hypothesis 2.0.1. Then $G \ncong$ $\mathrm{M^cL}$.*

Proof. ¿From [8, page 100], we see that $\mathrm{U}_4(3) \leq \mathrm{M^cL}$. Since $|\mathrm{U}_4(3)|_3 = |\mathrm{M^cL}|_3$ and we have shown in Lemma 2.2.3 that $\mathrm{U}_4(3)$ does not contain a Sylow 3-subgroup satisfying Hypothesis 2.0.1, we see that $\mathrm{M^cL}$ does not contain a Sylow 3-subgroup satisfying Hypothesis 2.0.1. Hence $G \ncong \mathrm{M^cL}$. □

Similarly, the next result follows from Corollary 2.3.2.

Corollary 2.3.3 *Suppose that G satisfies the conditions in Hypothesis 2.0.1. Then $G \ncong$ Co_2.*

Proof. We see that $\mathrm{M^cL} \leq \mathrm{Co}_2$ from [8, page 154]. Also $|\mathrm{Co}_2|_3 = |\mathrm{M^cL}|_3$. By Corollary 2.3.2, $\mathrm{M^cL}$ does not contain a Sylow 3-subgroup that satisfies Hypothesis 2.0.1, and hence Co_2 does not contain a Sylow 3-subgroup that satisfies Hypothesis 2.0.1. Hence $G \ncong$ Co_2. $\square$

So we have shown that G is not isomorphic to any of the known sporadic simple groups.

Proof (Proof of Theorem 2.0.2). This follows immediately from the $\mathcal{K}$-group hypothesis and the lemmas in Sections 2.1, 2.2 and 2.3. $\square$

Chapter 3

Some Strong Closure Results

In this chapter we prove three technical results, namely Theorems 3.3.3, 3.3.5 and 3.3.6, that will be used in the proof of Theorem B in Chapter 6. First we require some prerequisite definitions and background results.

3.1 Sym(4)-modules

We now present some results about GF(3) Sym(4)-modules.

Lemma 3.1.1 *Suppose that* $X \cong \mathrm{Sym}(4)$ *and that* V *is a faithful* 3*-dimensional* $\mathrm{GF}(3)X$*-module. Then:*

(i) there is a set of 1*-dimensional subspaces* $\mathcal{B} := \{\langle v_1\rangle, \langle v_2\rangle, \langle v_3\rangle\}$ *such that* $X/O_2(X)$ *acts as* $\mathrm{Sym}(3)$ *on* $\mathcal{B}$ *and each subspace in* $\mathcal{B}$ *is inverted by* $O_2(X)$*;*

(ii) X *has orbits of lengths* 3, 4 *and* 6 *on the* 1*-dimensional subspaces of* V *with representatives* $\langle v_1\rangle$, $\langle v_1+v_2+v_3\rangle$ *and* $\langle v_1+v_2\rangle$ *respectively; and*

(iii) X *has orbits of lengths* 3, 4, *and* 6 *on the* 2*-dimensional subspaces of* V *with representatives* $\langle v_1, v_2\rangle$, $\langle v_1+v_2, v_2+v_3\rangle$ *and* $\langle v_1, v_1+v_2+v_3\rangle$ *respectively.*

Proof. [29, Lemma 8] Suppose that $Q = O_2(X)$ and $Q^{\#} = \{q_1, q_2, q_3\}$. Since V is a faithful $\mathrm{GF}(3)X$-module and $\mathrm{Sym}(4)$ is not isomorphic to a subgroup of $\mathrm{GL}_2(3)$, we have that $C_V(Q) = \{0\}$. Therefore, Lemma 1.1.5 implies that

$$\begin{aligned} V &= \text{ the direct sum of } C_V(x) \text{ where } |Q : \langle x\rangle| = 2 \\ &= C_V(q_1) \oplus C_V(q_2) \oplus C_V(q_3). \end{aligned}$$

Also, since X acts transitively on $Q^{\#}$ by conjugation, X permutes the set of subspaces $\{C_V(q_i) \mid 1 \leq i \leq 3\}$ transitively and so each space has the same dimension. Let $\langle v_i\rangle = C_V(q_i)$, and we see that (i) holds.

Clearly $\{\langle v_i\rangle \mid 1 \leq i \leq 3\}$ forms an orbit of length 3 on the 1-dimensional subspaces of V. We also have that the subspaces $\langle v_1 \pm v_2 \pm v_3\rangle$ form an orbit of length 4 on V and the subspaces $\langle v_i \pm v_j\rangle$, where $i \neq j$ form an orbit of length 6, completing the proof of (ii).

Clearly the subspaces $\langle v_i, v_j\rangle$ for $i \neq j$ give an orbit of length 3 on the 2-dimensional subspaces of V. The subspaces $\langle v_i \pm v_j, v_j \pm v_k\rangle$ form an orbit of length 4. Finally the subspaces $\langle v_i, v_i + v_j - v_k\rangle$, for i, j and k distinct, form an orbit of length 6. We note that for each choice of i we have two choices for k. Hence (iii) holds. □

For the rest of this chapter whenever V is a faithful 3-dimensional $\mathrm{GF}(3)X$-module where $X \cong \mathrm{Sym}(4)$, we use basis $\mathcal{B} = \{v_1, v_2, v_3\}$ from Lemma 3.1.1.

Lemma 3.1.2 *Suppose that $X \cong \mathrm{Sym}(4)$ and that V is a faithful 3-dimensional $\mathrm{GF}(3)X$-module. Then the subspaces of order 3^2 are of the following types.*

Type 1 *These contain one 1-dimensional subspace conjugate to $\langle v_1 + v_2 + v_3\rangle$ and three 1-dimensional subspaces conjugate to $\langle v_1 + v_2\rangle$. There are four such subspaces.*

Type 2 *These contain two 1-dimensional subspaces conjugate to $\langle v_1 + v_2 + v_3\rangle$, one conjugate to $\langle v_1 + v_2\rangle$ and one conjugate to $\langle v_1\rangle$. There are six such subspaces.*

Type 3 These contains two 1*-dimensional subspaces conjugate to* $\langle v_1+v_2\rangle$ *and two conjugate to* $\langle v_1\rangle$*. There are three such subspaces.*

Proof. Let $W = \langle v_1+v_2, v_2-v_3\rangle$. Then W contains the 1-dimensional subspaces $\langle v_1+v_2\rangle$, $\langle v_2 - v_3\rangle$, $\langle v_1 + v_3\rangle$ and $\langle v_1 - v_2 - v_3\rangle$. Hence W is of type 1.

Now let $U = \langle v_1, v_1 + v_2 + v_3\rangle$. then U contains the 1-dimensional subspaces $\langle v_1\rangle$, $\langle v_1 + v_2 + v_3\rangle$, $\langle v_2 + v_3\rangle$ and $\langle v_1 - v_2 - v_3\rangle$. Hence U is of type 2.

Finally let $Y = \langle v_1, v_2\rangle$. Then Y contains the 1-dimensional subspaces $\langle v_1\rangle$, $\langle v_2\rangle$, $\langle v_1 + v_2\rangle$ and $\langle v_1 - v_2\rangle$. Hence Y is of type 3.

The number of each type of subgroup follows from Lemma 3.1.1 (*iii*). □

Lemma 3.1.3 *Suppose that* $X \cong \mathrm{Sym}(4)$ *and that* V *is a* 3*-dimensional faithful* $\mathrm{GF}(3)X$*-module. If* W *and* U *are two distinct subspaces of type 1 as described in Lemma 3.1.2, then* $W \cap U$ *is congruent to the* 1*-dimensional subspace* $\langle v_1 + v_2\rangle$*.*

Proof. The subspaces of V of type 1 are:

(*i*) $\langle v_1 + v_2, v_2 - v_3\rangle = \langle v_1 + v_2\rangle + \langle v_2 - v_3\rangle + \langle v_1 + v_3\rangle + \langle v_1 - v_2 - v_3\rangle$;

(*ii*) $\langle v_1 + v_2, v_1 - v_3\rangle = \langle v_1 + v_2\rangle + \langle v_1 - v_3\rangle + \langle v_2 + v_3\rangle + \langle -v_1 + v_2 - v_3\rangle$;

(*iii*) $\langle v_1 + v_3, v_1 - v_2\rangle = \langle v_1 + v_3\rangle + \langle v_1 - v_2\rangle + \langle v_2 + v_3\rangle + \langle -v_1 - v_2 + v_3\rangle$; and

(*iv*) $\langle v_1 - v_2, v_2 - v_3\rangle = \langle v_1 - v_2\rangle + \langle v_2 - v_3\rangle + \langle v_1 - v_3\rangle + \langle v_1 + v_2 + v_3\rangle$.

We see by inspection that the intersection of any two of these subspaces gives a subspace conjugate to $\langle v_1 + v_2\rangle$. □

We now define some notation for two different isomorphism types of a group of shape $3^3 : \mathrm{Sym}(4)$ in $\mathrm{Sym}(9)$.

Definition 3.1.4 Suppose that

$$A = \langle (123), (456), (789), (147)(258)(369), (14)(2536) \rangle.$$

Then A has shape $3^3 : \mathrm{Sym}(4)$ and $A \leq \mathrm{Alt}(9)$. We denote a group contained in $\mathrm{Sym}(9)$ which is isomorphic to A by $3^3 : \mathrm{Sym}(4)^+$. Similarly, if

$$B = \langle (123), (456), (789), (147)(258)(369), (14)(2536)(78) \rangle,$$

then B also has shape $3^3 : \mathrm{Sym}(4)$. We see that $B \leq \mathrm{Sym}(9)$ but $B \not\leq \mathrm{Alt}(9)$ and we denote a group contained in $\mathrm{Sym}(9)$ which is isomorphic to B by $3^3 : \mathrm{Sym}(4)^-$.

We note that for a set $\Omega = \{a_1, a_2, \ldots, a_n\}$, $\mathrm{Sym}(\{a_1, a_2, \ldots, a_n\})$ denotes the group of permutations of the elements of Ω and $\mathrm{Alt}(\{a_1, a_2, \ldots, a_n\})$ denotes the group of even permutations.

Lemma 3.1.5 *Suppose that H is a group such that $O_3(H)$ is elementary abelian of order 3^3, $H/O_3(H) \cong \mathrm{Sym}(4)$ and $C_H(O_3(H)) = O_3(H)$. Then H embeds into $\mathrm{Sym}(9)$ and H is isomorphic to either $3^3 : \mathrm{Sym}(4)^+$ or $3^3 : \mathrm{Sym}(4)^-$.*

Proof. Clearly the groups $3^3 : \mathrm{Sym}(4)^+$ and $3^3 : \mathrm{Sym}(4)^-$ satisfy the hypothesis. Suppose that H satisfies the hypothesis and let $V = O_3(H)$. Since $C_H(V) = V$, we see that V is a faithful 3-dimensional $H/V \cong \mathrm{Sym}(4)$-module. Therefore V can be identified with a module as in Lemma 3.1.1. Let v_i and q_i be as in the proof of Lemma 3.1.1 and $K = \langle v_1, v_2, q_1, q_2, t \rangle$ where $q_1^t = q_2$ and so t is an involution. Then $S = \langle q_1, q_2, t \rangle \in \mathrm{Syl}_2(H)$. Since $|K| = 9.8$, we see that K has index 9 in H and $VS > K$ with $|VS : K| = 3$. Since $\bigcap_{h \in H} K^h = 1$, we have that H has a faithful permutation representation of degree 9 and therefore H embeds into $\mathrm{Sym}(9)$. This representation preserves blocks of size 3. Therefore, H embeds into $\mathrm{Sym}(3) \wr \mathrm{Sym}(3)$. Let $L \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(3)$. Then H is a normal subgroup of L of index 2. So $V \lhd L$ and $L/V \cong 2 \times \mathrm{Sym}(4)$ and therefore H/V is a normal subgroup of index 2 in L/V and there are three possibilities to consider for H/V.

(*i*) $\langle(12),(345),(456)\rangle \cong 2 \times \mathrm{Alt}(4)$: If $T \in \mathrm{Syl}_2(2 \times \mathrm{Alt}(4))$, then T is elementary abelian. Clearly S is not elementary abelian and hence $H/V \ncong \mathrm{Sym}(4)$. Therefore, we can eliminate this case.

(*ii*) $\langle(12)(34),(2)(45),(12)(56)\rangle \cong \mathrm{Sym}(4)$: This case gives us one isomorphism type of the group $3^3 : \mathrm{Sym}(4)$ that is contained in $\mathrm{Sym}(9)$. In addition this group is contained in $\mathrm{Alt}(9)$ and so we have the group $3^3 : \mathrm{Sym}(4)^+$.

(*iii*) $\langle(34),(45),(56)\rangle \cong \mathrm{Sym}(4)$: This case gives us a second isomorphism type of $3^3 : \mathrm{Sym}(4)$ contained in $\mathrm{Sym}(9)$. Clearly this case is distinct from (*ii*) since it is not contained in $\mathrm{Alt}(9)$ and so we have the group $3^3 : \mathrm{Sym}(4)^-$.

Therefore there are two isomorphism types of $3^3 : \mathrm{Sym}(4)$ in $\mathrm{Sym}(9)$, one of which is contained in $\mathrm{Alt}(9)$. $\square$

Theorem 3.1.6 *Suppose that* $X \cong H \in \{3^3 : \mathrm{Sym}(4)^+, 3^3 : \mathrm{Sym}(4)^-\}$ *and that* $x \in X \backslash X'$ *has order two. If* $|C_X(x)| = 2^2 3$*, then* $X \cong 3^3 : \mathrm{Sym}(4)^+$.

Proof. Suppose $X \cong H \in \{3^3 : \mathrm{Sym}(4)^+, 3^3 : \mathrm{Sym}(4)^-\}$ and $x \in X$ has order 2. Then either $x \in X'$ and hence x is conjugate to $(12)(45)$, or $x \in X \backslash X'$. Suppose $x \in X \backslash X'$ and $H = 3^3 : \mathrm{Sym}(4)^-$. Then x is conjugate to $(14)(25)(36)$ and

$$|C_X(x)| = |\langle(123)(456),(789),(14)(25)(36),(12)(45)\rangle| = 2^2 3^2.$$

Now suppose $H = 3^3 : \mathrm{Sym}(4)^+$ and $x \in X \backslash X'$. Then x is conjugate to $(14)(25)(36)(78)$ and

$$|C_X(x)| = |\langle(123)(456),(14)(25)(36)(78),(12)(45)\rangle| = 2^2 3.$$

Therefore, if $|C_X(x)| = 2^2 3$, then $X \cong 3^3 : \mathrm{Sym}(4)^+$. $\square$

3.2 Some Results for $\mathrm{Alt}(9)$ and $3^3 : \mathrm{Sym}(4)^+$

We now prove a number of results concerning the groups $\mathrm{Alt}(9)$ and $3^3 : \mathrm{Sym}(4)^+$, where $3^3 : \mathrm{Sym}(4)^+$ is as defined in Definition 3.1.4. First we require some notation for the conjugacy classes of $\mathrm{Alt}(9)$ and $3^3 : \mathrm{Sym}(4)^+$.

Notation 3.2.1 Suppose that $X \cong \mathrm{Alt}(9)$. Then:

(*i*) elements conjugate in X to $(12)(34)$ are said to be in class 2A;

(*ii*) elements conjugate in X to $(12)(34)(56)(78)$ are said to be in class 2B;

(*iii*) elements conjugate in X to (123) are said to be in class 3A;

(*iv*) elements conjugate in X to $(123)(456)(789)$ are said to be in class 3B; and

(*v*) elements conjugate in X to $(123)(456)$ are said to be in class 3C.

We note that this notation corresponds to Atlas (see [8]) notation.

Now suppose that $X \cong 3^3 : \mathrm{Sym}(4)^+$. Then:

(*i*) elements conjugate in X to $(12)(45)$ are said to be in class 2A;

(*ii*) elements conjugate in X to $(14)(25)(36)(78)$ are said to be in class 2B; and

(*iii*) elements conjugate in X to (123) are said to be in class 3A.

We see that these definitions of conjugacy classes correspond to the conjugacy classes of elements of order 2 and the elements of class 3A in $\mathrm{Alt}(9)$ and we note that all elements in X of cycle shape 3 are in X-conjugacy class 3A.

Let $a = (14)(25)(36)(78)$ and $b = (45)(78)$. Consider the subgroup $S = \langle a, b\rangle$. Since $ab = (1524)(36)$ has order 4, we see that $S \cong \mathrm{Dih}(8)$ and therefore $S \in \mathrm{Syl}_2(3^3 : \mathrm{Sym}(4)^+)$. We note that $Z(S) = \langle (12)(45)\rangle$.

Lemma 3.2.2 *Suppose that $X \cong \mathrm{Alt}(9)$. Then the following hold.*

(i) *If $x \in X$ is in class 2A, then $|C_X(x)| = 2^5 3.5$ and if $x \in X$ is in class 2B, then $|C_X(x)| = 2^6 3$.*

(ii) *If $x \in X$ is in class 2A and $E \in \mathrm{Syl}_3(C_X(x))$, then the non-trivial elements of E are in class 3A and if $x \in X$ is in class 2B and $E \in \mathrm{Syl}_3(C_X(x))$, then the non-trivial elements of E are in class 3C.*

(iii) *If $x \in X$ is in class 3A, then $|C_X(x)| = 2^3 3^3 5$, if $x \in X$ is in class 3B, then $|C_X(x)| = 3^4$, and if $x \in X$ is in class 3C, then $|C_X(x)| = 2.3^3$. If $x \in X$ has order 3, then $C_X(\langle x \rangle)$ has index two in $N_X(\langle x \rangle)$.*

Proof. (i) Let $x = (12)(34)$ and $y = (12)(34)(56)(78)$. So we see from Notation 3.2.1, that x and y are in classes 2A and 2B respectively and we have that

$$C_X(x) = \langle \mathrm{Alt}(\{5,6,7,8,9\}), (12)(34), (13)(24), (12)(56) \rangle,$$

and so $|C_X(x)| = 2^3 |\mathrm{Alt}(5)| = 2^5 3.5$. Similarly,

$$C_X(y) = \langle (12)(34), (34)(56), (56)(78), \mathrm{Sym}(\{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}) \rangle,$$

and so $|C_X(y)| = 2^4 |\mathrm{Sym}(4)| = 2^6 3$.

(ii) We see from the proof of (i) that if $x \in X$ has order 2 and $E \in \mathrm{Syl}_3(C_X(x))$, then $|E| = 3$. So, E is generated by an element of order 3, y say. We see that if x is in class 2A then y is conjugate to (567) and hence is in class 3A. Similarly, if x is in class 2B then y is conjugate to (135)(246) and hence is in class 3C.

(iii) Let $a = (123)$, $b = (123)(456)(789)$ and $c = (123)(456)$. So we see from Notation

3.2.1, that a, b and c are in classes 3A, 3B and 3C respectively and we have that

$$C_X(a) = \langle \mathrm{Alt}(\{4,5,6,7,8,9\}), (123) \rangle.$$

So $|C_X(a)| = 3|\,\mathrm{Alt}(6)| = 2^3 3^3 5$. Similarly,

$$C_X(b) = \langle (123), (456), (789), (147)(258)(369) \rangle$$

and

$$C_X(c) = \langle (789), (123), (456), (14)(25)(36) \rangle.$$

Therefore $|C_X(b)| = 3^4$ and $|C_X(c)| = 2.3^3$.

In each case we can find an element in X that normalizes, but does not centralize $\langle x \rangle$ where $x \in X$ has order 3. For example, the element (12)(35) normalizes but does not centralize $\langle a \rangle$. Hence $|N_X(\langle x \rangle) : C_X(\langle x \rangle)| = 2$ for all $x \in X$ of order 3. $\square$

Lemma 3.2.3 *Suppose that* $X \cong 3^3 : \mathrm{Sym}(4)^+$. *If* $x \in X$ *is in class 2A, then* $|C_X(x)| = 2^3 3$ *and if* $x \in X$ *is in class 2B, then* $|C_X(x)| = 2^2 3$.

Proof. Let $x = (12)(45)$ and $y = (14)(25)(36)(78)$ be representatives from classes 2A and 2B respectively. Then

$$C_X(x) = \langle (789), (12)(45), (14)(25)(36)(78), (45)(78) \rangle,$$

and so $|C_X(x)| = 2^3 3$. We see from the proof of Theorem 3.1.6 that

$$C_X(y) = \langle (123)(456), (12)(45), (14)(25)(36)(78) \rangle,$$

and hence $|C_X(y)| = 2^2 3$. $\square$

We conclude this section with some results concerning a particular type of GF(2)X-module where $X \cong 3^3 : \mathrm{Sym}(4)^+$ or $X \cong \mathrm{Alt}(9)$.

Lemma 3.2.4 *Suppose that $X \cong 3^3 : \mathrm{Sym}(4)^+$ and that V is a* GF(2)X*-module of dimension 8 such that the elements in X-conjugacy class 3A act fixed-point-freely on V. Let*

$$S = \langle (14)(25)(36)(78), (45)(78) \rangle \in \mathrm{Syl}_2(X).$$

Then the following hold.

(i) If $x \in X$ has order 2, then $|C_V(x)| = 2^4$. In particular $C_V(x) = [V, x]$.

(ii) If $F \leq X$ has order 4, then $|C_V(F)| = 2^2$.

(iii) $|C_V(S)| = 2$.

Proof. (i) We notice first that if $x \in X$ has order 2, then $|C_V(x)| \geq 2^4$ by Lemma 1.3.1 (ii). Suppose that either $a = (23)(45)$ and $b = (13)(45)$ or $a = (14)(25)(36)(78)$ and $b = (14)(25)(36)(89)$. So a and b are in the same conjugacy class of X. In either case ab is a 3-cycle, and hence is in class 3A. We have that $C_V(a) \cap C_V(b) \leq C_V(ab) = \{0\}$ since the 3-cycles of G act fixed-point-freely on V. So,

$$\begin{aligned} 8 &= \dim V \\ &\geq \dim(C_V(a) + C_V(b)) \\ &= \dim C_V(a) + \dim C_V(b) - \dim(C_V(a) \cap C_V(b)) \\ &\geq 4 + 4 + 0. \end{aligned}$$

Therefore $\dim C_V(a) = \dim C_V(b) = 4$ and hence if x has order 2, $|C_V(x)| = 2^4$. By Lemma 1.3.1, $V/C_V(x) \cong [V, x]$ and $C_V(x) \geq [V, x]$. Since $|V| = 2^8$, this implies that $|V/C_V(x)| = 2^4 = |[V, x]|$. Hence $C_V(x) = [V, x]$.

(ii) We see from Notation 3.2.1 that $S \in \mathrm{Syl}_2(X)$. Also, $Z(S) = \langle x \rangle$ where $x = (12)(45)$. Suppose that $F \leq S$ such that $|F| = 4$. Then F is abelian and normal in S, so $F \geq Z(S)$ and hence, $C_V(F) \leq C_V(x)$. We have three possibilities for F:

$$F = \begin{cases} \langle (12)(45), (45)(78) \rangle; \\ \langle (12)(45), (14)(25)(36)(78) \rangle; \text{ or} \\ \langle (1524)(36) \rangle. \end{cases}$$

Suppose that $e \in C_X(x)$. Since v and e commute, where $v \in C_V(x)$, we have,

$$(v^e)^x = v^{ex} = v^{xe} = v^e,$$

and so $v^e \in C_V(x)$. Hence, $C_V(x)$ admits an action on $C_X(x)$.

We have that $F/\langle x \rangle$ acts as, at most, an involution on $C_V(x)$. Therefore, $|C_V(F)| = |C_{C_V(x)}(F)| \geq 2^2$ by Lemma 1.3.1 (ii).

Suppose that $d = (789)$ and F is a fours group. Then an easy calculation shows that $\langle F, F^d \rangle \geq \langle d \rangle$. Therefore, since $C_V(d) = \{0\}$, $\dim(C_{C_V(x)}(F) \cap C_{C_V(x)}(F^d)) = 0$ and so,

$$\begin{aligned} 4 &= \dim C_V(x) \\ &\geq \dim(C_V(F) + C_V(F^d)) \\ &= \dim C_V(F) + \dim C_V(F^d) - \dim(C_V(F) \cap C_V(F^d)) \\ &= \dim C_V(F) + \dim C_V(F^d) \\ &\geq 2 + 2. \end{aligned}$$

Hence $|C_V(F)| = 2^2$.

Now suppose that F is cyclic. Hence $F = \langle c \rangle$, where $c = (1524)(36)$. Since c and d commute, $C_{C_V(x)}(F) = C_V(F)$ is $\langle d \rangle$-invariant. Therefore, as $2^4 \geq |C_V(F)| \geq 2^2$ and $2^3 \equiv -1 \mod 3$, we have that $|C_V(F)| \in \{2^2, 2^4\}$. Suppose that $|C_V(F)| = 2^4$. Then $[V, x, c] = 0$. Also, since x and c commute, $[x, c, V] = [1, V] = 0$ and hence $[c, V, x] = [V, c, x] = 0$ by the Three Subgroup Lemma. So $[V, c] \leq C_V(x) = C_V(F)$. Hence $[V, c, c] = 0$. So, we have that

$$0 = (v^c - v)^c - (v^c - v) = v^{c^2} - v^c - v^c + v = v^{c^2} + v,$$

since V is a vector space defined over GF(2). Therefore, $v^{c^2} = v$ for all $v \in V$. So, as $x = c^2$, $v^x = v$ and therefore $C_V(x) = V$, a contradiction. Hence $|C_V(F)| = 2^2$, if F is cyclic.

(*iii*) If F is a fours group then, since F inverts d, if $C_V(F)$ is d-invariant, then $|C_V(d)| \neq 1$, a contradiction. So $C_V(S) \leq C_V(C) \cap C_V(F)$, where C is a cyclic subgroup of S of order 4. and hence, as $|C_V(S)| \neq 1$, $|C_V(S)| = 2$. $\square$

Lemma 3.2.5 *Let $X \cong 3^3 : \mathrm{Sym}(4)^+$. Suppose that V is a faithful* GF(2)X*-module of dimension* 8 *such that the elements in X-conjugacy class 3A act fixed-point-freely on V. Then V is irreducible.*

Proof. Let W be a proper submodule of V. Then W is a faithful GF(2)X-module since the elements of X-conjugacy class act fixed-point-freely on V. So $\dim W \geq 6$ by comparing $|\mathrm{GL}_5(2)|_3$ and $|X|_3$. Also, V/W is a faithful GF(2)X-module by Coprime Action. Hence $\dim V/W \geq 6$, since W is a proper submodule of V. Therefore $\dim V \geq 12$, which is impossible. Therefore V is irreducible. $\square$

Lemma 3.2.6 *Suppose that $X \cong \mathrm{Alt}(9)$ and V is a $\mathrm{GF}(2)X$-module of dimension 8 such that the elements in X-conjugacy class 3A act fixed-point-freely on V. Then the following hold.*

(i) If $x \in X$ has order 2, then $|C_V(x)| = 2^4$. In particular, $C_V(x) = [V, x]$.

(ii) If $x \in X$ is in class 3A, then $|C_V(x)| = 1$, if $x \in X$ is in class 3B, then $|C_V(x)| = 2^2$ and if $x \in X$ is in class 3C, then $|C_V(x)| = 2^4$.

(iii) Suppose that $x \in X$ is in class 2B and that $E \in \mathrm{Syl}_2(C_X(x))$. Then $[C_V(x), E] \neq 1$.

(iv) X has two orbits on $V^{\#}$. One orbit has length 135 and if v is a representative from this orbit, $C_X(v) \sim 2^3 : \mathrm{L}_3(2)$. The other has length 120 and if w is a representative, then $C_X(w) \sim \mathrm{L}_2(8) : 3$.

Proof. (*i*) We notice first that if $x \in X$ has order 2, then $|C_V(x)| \geq 2^4$ by Lemma 1.3.2. Let either: $a = (23)(45)$ and $b = (13)(45)$; or $a = (12)(34)(56)(78)$ and $b = (12)(34)(56)(89)$. So a and b are in the same conjugacy class of X. In either case ab is a 3-cycle, and hence is in class 3A. We have that $C_V(a) \cap C_V(b) \leq C_V(ab) = \{0\}$ since the 3-cycles of X act fixed-point-freely on V. So,

$$\begin{aligned} 8 &= \dim V \\ &\geq \dim(C_V(a) + C_V(b)) \\ &= \dim C_V(a) + \dim C_V(b) - \dim(C_V(a) \cap C_V(b)) \\ &\geq 4 + 4 + 0. \end{aligned}$$

Therefore, since a and b are X-conjugate, $\dim C_V(a) = \dim C_V(b) = 4$ and hence if $x \in G$ has order 2, $|C_V(x)| = 2^4$. By Lemma 1.3.1, $V/C_V(x) \cong [V, x]$ and $C_V(x) \geq [V, x]$. Since $|V| = 2^8$, this implies that $|V/C_V(x)| = 2^4 = |[V, x]|$. Hence $C_V(x) = [V, x]$.

(*ii*) Let $a = (123)$, $b = (123)(456)(789)$ and $c = (123)(456)$. So a, b and c are representatives of classes 3A, 3B and 3C respectively. Since the 3-cycles of X act fixed-point-freely on V by assumption we see that $|C_V(a)| = 1$.

Let $A = \langle (123), (456) \rangle$. Since $C_V(A) \leq C_V((123)) = \{0\}$, by Lemma 1.1.5,

$$\begin{aligned} V &= \text{ the direct sum of } C_V(x) \text{ where } |A : \langle x \rangle| = 3 \\ &= C_V((123)) \oplus C_V((456)) \oplus C_V((123)(456)) \oplus C_V((132)(456)). \end{aligned}$$

We have that $C_V((123)) = C_V((456)) = \{0\}$ by assumption and

$$\dim(C_V((123)(456))) = \dim(C_V((132)(456))),$$

since $(123)(456)$ and $(132)(456)$ are in the same conjugacy class of X. Hence $|C_V(c)| = 2^4$.

Now let $B = \langle (123), (456)(789) \rangle$. Again, since $C_V(B) \leq C_V((123)) = \{0\}$, by Lemma 1.1.5 we have that,

$$\begin{aligned} V &= \text{ the direct sum of } C_V(x) \text{ where } |B : \langle x \rangle| = 3 \\ &= C_V((123)) \oplus C_V((456)(789)) \oplus C_V((123)(456)(789)) \oplus C_V((132)(456)(789)). \end{aligned}$$

We have that $|C_V((123))| = 1$ by assumption and $|C_V((456)(789))| = 2^4$ by above. Hence, since $\dim(C_V((123)(456)(789))) = \dim(C_V((132)(456)(789)))$, we see that $|C_V(b)| = 2^2$.

(*iii*) Let $x \in X$ be in class 2B and $E \in \mathrm{Syl}_3(C_X(x))$. Suppose $[C_V(x), E] = 1$. We have that $[E, x] = 1$ since $E \leq C_X(x)$. Therefore, by Thompson's $A \times B$ Lemma, $[V, E] = 0$. Hence, $C_V(E) = 1$. This is a contradiction since the non-trivial elements of E are in class 3C by Lemma 3.2.2 (ii) and so $|C_V(E)| = 2^4$ by (ii). Hence, $[C_V(x), E] \neq 1$ as required.

(iv) We note that $|V^\#| = 2^8 - 1 = 255$. Let $T \in \mathrm{Syl}_2(X)$. If $T_0 = V : T$, then $V \trianglelefteq T_0$, and so $Z(T_0) \cap V \neq 1$ by Lemma 1.1.12. Therefore $C_V(T) \neq \{0\}$. So suppose that $v \in C_V(T)$ is non-zero. Then $C_X(v) \geq T$ which implies that $C_X(v) \leq M$ where $M \cong \mathrm{Alt}(8)$, since the nine subgroups of Alt(9) that are isomorphic to Alt(8) are the only maximal subgroups of Alt(9) of odd index, see [8, page 37]. Since $T \in \mathrm{Syl}_2(\mathrm{Alt}(8))$, we have that $|M : C_X(v)|$ must be odd. Therefore, $C_X(v) \leq A$, where A is isomorphic to a maximal subgroup of $M \cong \mathrm{Alt}(8)$ of odd index. So using [8, page 22], we see that A is isomorphic to one of:

(a) Alt(8) of index 1;

(b) $2^3 : \mathrm{L}_3(2)$ of index 15; or

(c) $2^4 : (\mathrm{Sym}(3) \times \mathrm{Sym}(3))$ of index 35.

Since M is a subgroup of X isomorphic to Alt(8), M certainly contains 3-cycles and $C_V(d) = \{0\}$ where d is a 3-cycle in M by assumption. Therefore $\{0\} = C_V(d) \supseteq C_V(M)$ and $C_V(M) = \{0\}$, a contradiction. So $C_X(v) \ncong \mathrm{Alt}(8)$. So, by the Orbit-Stabilizer Theorem, either $9 \cdot 15$ or $9 \cdot 35$ divide $|v^X|$. However $9 \cdot 35 = 315 \geq 255$ and $2(9 \cdot 15) = 270 \geq 255$. Hence the only possibility is that $|v^X| = 9 \cdot 15 = 135$ and $C_X(v) \sim 2^3 : \mathrm{L}_3(2)$. So we have $255 - 135 = 120$ vectors left.

Since $C_X(v)$ is contained in a subgroup of X isomorphic to Alt(8), we see that there exists an element of X which does not fix any member of v^X. For example the element $b = (123)(456)(789)$. Since $|C_V(b)| = 2^2$ by (ii), we can choose a non-zero $y \in C_V(b)$. So $C_X(y) \neq C_X(v)$ since b does not centralize v and so we see that $C_X(y)$ is contained in a maximal subgroup of X of index 120 or less. So, $C_X(y)$ is isomorphic to a subgroup of:

(a) Alt(8);

(b) Sym(7);

(c) $(3 \times \mathrm{Alt}(6)) : 2$; or

(d) $\mathrm{L}_2(8) : 3$

Since b is not contained in a subgroup of Alt(9) which is isomorphic to Alt(8) or Sym(7), we see that $C_X(y) \not\leq N$ where N is isomorphic to Alt(8) or Sym(7). Suppose $C_X(y)$ is isomorphic to a subgroup of $(3 \times \mathrm{Alt}(6)) : 2$. Then since the 3-cycles of X act fixed-point-freely on V and $3 \times \mathrm{Alt}(6) \cong (123) \times \mathrm{Alt}(\{4, 5, 6, 7, 8, 9\}$, this implies that $C_X(y) \leq N$ where $N \cong \mathrm{Alt}(6) : 2 \cong \mathrm{Sym}(6)$. Since b is not contained in a subgroup of Alt(9) isomorphic to Sym(6), this is a contradiction. So $C_X(y) \leq N$ where $N \cong \mathrm{L}_2(8) : 3$ and since $|X : (\mathrm{L}_2(8) : 3)| = 120$, this implies that $C_X(y) \sim \mathrm{L}_2(8) : 3$ and $|y^X| = 120$. We have now exhausted the vectors in V and hence the result holds. □

3.3 The Theorems

In order to prove Theorem 3.3.3 we first require a definition and a preliminary lemma.

Definition 3.3.1 Suppose G is a group, $R \leq G$ and $L = N_G(R)$. Assume that $L/R = K$ where $K \cong \mathrm{Alt}(9)$. Let $g \in L$ have order 3. Then:

(i) if gR is in K-conjugacy class 3A, define $\mathcal{A} = \{g^G\}$;

(ii) if gR is in K-conjugacy class 3B, define $\mathcal{B} = \{g^G\}$; and

(iii) if gR is in K-conjugacy class 3C, define $\mathcal{C} = \{g^G\}$.

We note that it is not obvious that Definition 3.3.1 is well defined. However, we only use it in situations where it is.

Lemma 3.3.2 *Let G be a group and $L \leq G$. For $g \in L$ define $\mathcal{X} = \{g^G\}$. Suppose that whenever $g \in \mathcal{X} \cap L$ then $N_G(\langle g \rangle) \leq L$ and that $x \in G$. If $g \in \mathcal{X} \cap L^x$, then $N_G(\langle g \rangle) \leq L^x$.*

Proof. For $x \in G$, if $g \in \mathcal{X} \cap L^x$, then $g^{x^{-1}} \in \mathcal{X} \cap L$. Hence $N_G(\langle g^{x^{-1}} \rangle) \leq L$ and therefore $N_{G^x}(\langle g \rangle) \leq L^x$. However, since $x \in G$, we have that $N_G(\langle g \rangle) \leq L^x$ as required. $\square$

Theorem 3.3.3 *Suppose that G is a group, $R \leq G$ and $L = N_G(R)$. Assume that:*

(i) *$L/R = K$, where $K \cong \mathrm{Alt}(9)$, R is elementary abelian of order 2^8 and R is the unique minimal normal subgroup of L;*

(ii) *the elements of $\mathcal{A} \cap L$ act fixed-point-freely on R;*

(iii) *the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are disjoint; and*

(iv) *if $g \in \mathcal{B} \cap L$ or $g \in \mathcal{C} \cap L$, then $N_G(\langle g \rangle) \leq L$.*

Then R is strongly closed in L with respect to G.

Proof. We first note that by assumptions (i) and (ii), R may be considered as a $\mathrm{GF}(2)K$-module of dimension 8 such that the elements in K-conjugacy class 3A act fixed-point-freely on R. So we may use the results from Lemma 3.2.6.

Suppose that R is not strongly closed in L. Let $S \in \mathrm{Syl}_2(L)$ and let R^x be a conjugate of R in G such that we may choose $r \in (R^x \cap S) \backslash R$.

Since $r \notin R$, we have that $C_R(r) = [R, r]$ by Lemma 3.2.6 (i). Therefore, Lemma 1.1.18 implies that, $|C_L(r)| = |C_R(r)||C_{L/R}(rR)| = |C_R(r)||C_K(rR)|$. Also $rR \in K$ has order 2, and hence by Lemmas 3.2.2 (i) and 3.2.6 (i), either:

A. $|C_L(r)| = 2^4 2^5 3.5$; or

B. $|C_L(r)| = 2^4 2^6 3$.

The Sylow 3-subgroups have order 3 in both cases and by Lemma 3.2.2 (ii), $\mathcal{A}\cap C_L(r) \neq \emptyset$ and $\mathcal{C} \cap C_L(r) \neq \emptyset$ in cases A and B respectively.

We have that $C_{L^x}(r)/R^x = C_{L^x/R^x}(r)$, $L^x/R^x \cong K$ and $r \in R^x$. Hence, by Lemma 3.2.6 (iv), either:

C. $C_{L^x}(r) \sim 2^8.2^3 : \mathrm{L}_3(2)$ and the Sylow 3-subgroups of $C_{L^x}(r)$ have order 3; or

D. $C_{L^x}(r) \sim 2^8.\mathrm{L}_2(8) : 3$ and the Sylow 3-subgroups of $C_{L^x}(r)$ are extra-special of order 27.

By assumption, the elements of $\mathcal{A}\cap L$ act fixed-point-freely on R. Hence, the elements of $\mathcal{A}\cap L^x$ act fixed-point-freely on R^x. So $\mathcal{A}\cap C_{L^x}(r) \neq \emptyset$. Hence, if $g \in C_{L^x}(r)$ has order 3, then $g \in \mathcal{B}$ or $g \in \mathcal{C}$.

Let $T \in \mathrm{Syl}_3(C_{L^x}(r))$. We claim that $T \in \mathrm{Syl}_3(C_G(r))$. Suppose that $T \notin \mathrm{Syl}_3(C_G(r))$. Then $N_{C_G(r)}(T) \nleq C_{L^x}(r)$ and so $N_{C_G(r)}(T) \nleq L^x$. However assumption (iv) and Lemma 3.3.2 imply that $N_{C_G(r)}(T) \leq N_{C_G(r)}(Z(T)) \leq L^x$, and hence this is a contradiction since $Z(T) \subseteq \mathcal{B}$ or $Z(T) \subseteq \mathcal{C}$. Hence $\mathrm{Syl}_3(C_{L^x}(r)) \subseteq \mathrm{Syl}_3(C_G(r))$.

So, $C_G(r)$ does not contain any elements of $\mathcal{A}$. Hence, $C_L(r)$ cannot contain any elements from $\mathcal{A}$ otherwise $\mathcal{A}$ would have a non-trivial intersection with either $\mathcal{B}$ or $\mathcal{C}$, contradicting assumption (iii). Hence case A cannot occur and so $|C_L(r)| = 2^4 2^6 3$ and $\mathcal{C} \cap C_L(r) \neq \emptyset$.

So, let $T_L \in \mathrm{Syl}_3(C_L(r))$. We claim that $T_L \in \mathrm{Syl}_3(C_G(r))$. Suppose that $T_L \notin \mathrm{Syl}_3(C_G(r))$. Then similarly to above, $N_{C_G(r)}(T_L) \nleq C_L(r)$ and hence $N_{C_G(r)}(T_L) \nleq L$. However, the non-trivial elements of T_L are in $\mathcal{C}$. So by assumption (iv), $N_{C_G(r)}(T_L) \leq L$ and we have a contradiction. Hence $\mathrm{Syl}_3(C_L(r)) \subseteq \mathrm{Syl}_3(C_G(r))$.

In particular, the Sylow 3-subgroups of $C_L(r)$ and $C_{L^x}(r)$ are $C_G(r)$-conjugate. Therefore, we see that cases B and C occur.

Let $D \in \mathrm{Syl}_3(C_L(r))$ and $D_1 \in \mathrm{Syl}_3(C_{L^x}(r))$. We have that $C_{L^x}(r)/R^x = C_{L^x/R^x}(r)$,

$L^x/R^x \cong K$ and $r \in R^x$. Since D_1 is $C_G(r)$-conjugate to D, we have that D_1 is generated by an element of K-conjugacy class 3C. Hence, by Lemmas 3.2.2 (*iii*) and 3.2.6 (*ii*), we have that

$$|C_{C_{L^x(r)}}(D_1)| = |C_{R^x}(D_1)||C_{C_K(r)}(D_1)| = 2^4 2.3.$$

Now $|C_{C_L(r)}(D)| = |C_{C_R(r)}(D)||C_{C_K(r)}(D)| = 2^a 2.3$ where $a \leq 4$ by Lemma 3.2.2 (*iii*). Suppose that $|C_{C_R(r)}(D)| = 2^4$. Then $C_R(r) \leq C_R(D)$. However, since $[C_R(r), D] \neq 0$ by Lemma 3.2.6 (*iii*), this is a contradiction. So, $C_{C_R(r)}(D) < 2^4$. Since $D \subseteq \mathcal{C}$, we have that $C_G(D) \leq L$, and hence $C_{C_G(r)}(D) \leq L$. Therefore $C_{C_G(r)}(D) = C_{C_L(r)}(D)$. Also $|C_{C_G(r)}(D_1)| = |C_{C_G(r)}(D)|$ since D and D_1 are $C_G(r)$-conjugate. Therefore we have that $2^5 = |C_{C_{L^x(r)}}(D_1)|_2 \leq |C_{C_G(r)}(D_1)|_2 = |C_{C_L(r)}(D)|_2 < 2^5$, which is a contradiction. Hence R is strongly closed in L with respect to G. □

We prove a similar result for $K \cong 3^3 : \mathrm{Sym}(4)^+$ in two steps.

Lemma 3.3.4 *Suppose that G is a group, $R \leq G$ and $L = N_G(R)$. Assume that:*

(i) $L/R = K$, where $K \cong 3^3 : \mathrm{Sym}(4)^+$ and R is elementary abelian of order 2^8; and

(ii) the elements of K-conjugacy class 3A act fixed-point-freely on R.

Then $R = J(S)$, where $S \in \mathrm{Syl}_2(L)$. In particular, $syl_2(L) \subseteq \mathrm{Syl}_2(G)$.

Proof. We first note that the assumptions imply that R may be regarded as a GF(2)K-module of dimension 8 such that the elements of K-conjugacy class 3A act fixed-point-freely on R and hence we may use the results from Lemma 3.2.4.

Let $S \in \mathrm{Syl}_2(L)$. We claim $R = J(S)$. Recall from Definition 1.1.9 that

$$J(S) = \langle A \mid A \in \mathcal{A}(S) \rangle,$$

where $\mathcal{A}(S)$ is the set of abelian subgroups of S of maximal order. Suppose $R \neq J(S)$ and let $F \in \mathcal{A}(S)$ with $F \neq R$. So $F \leq S$ is an abelian 2-group with $|F| \geq 2^8$. We have that FR/R is a 2-subgroup of $L/R = K \cong 3^3 : \mathrm{Sym}(4)^+$. Therefore $|FR/R| \leq |3^3 : \mathrm{Sym}(4)^+|_2 = 2^3$. Since $FR/R \cong F/(F \cap R)$, this implies that $|F \cap R| \geq 2^5$. We have that F is abelian, and hence, $F \cap R \leq C_R(F)$. However, since FR/R contains an element of order 2, by Lemma 3.2.4 (i),

$$2^5 \leq |R \cap F| \leq C_R(F) \leq 2^4,$$

which is absurd and so $R = F$. Hence $R = J(S)$. □

Theorem 3.3.5 *Suppose that G is a group, $R \leq G$ and $L = N_G(R)$. Assume that:*

(i) $L/R = K$, where $K \cong 3^3 : \mathrm{Sym}(4)^+$ and R is elementary abelian of order 2^8; and

(ii) the elements of K-conjugacy class 3A act fixed-point-freely on R;

Then R is strongly closed in L with respect to G.

Proof. We note that as in the proof of Lemma 3.3.4, the assumptions imply that we may use the results from Lemma 3.2.4.

Suppose R is not strongly closed in L. Let $S \in \mathrm{Syl}_2(L)$ and let $R^\dagger$ be a conjugate of R in G such that we may choose $r \in (R^\dagger \cap S)\backslash R$.

Since $r \notin R$, we have that $C_R(r) = [R, r]$ by Lemma 3.2.4 (i). Therefore, Lemma 1.1.18 implies that, $|C_L(r)| = |C_R(r)||C_{L/R}(rR)| = |C_R(r)||C_K(rR)|$. Hence, by Lemmas 3.2.3 and 3.2.4 (i), since rR is in K, either:

A. $|C_L(r)| = 2^4 2^3 3$ and $|C_S(r)| = 2^7$; or

B. $|C_L(r)| = 2^4 2^2 3$ and $|C_S(r)| = 2^6$.

Hence $|C_S(r)R/R| = 8$ or 4 in cases A and B respectively. Since $C_R(r)R/R \leq K \cong 3^3 :$

$\mathrm{Sym}(4)^+$, Lemma 3.2.4 (ii) and (iii) imply that

$$|C_R(C_S(r))| = \begin{cases} 2, & \text{if } |C_S(r)R/R| = 8; \\ 2^2, & \text{if } |C_S(r)R/R| = 4. \end{cases} \tag{3.1}$$

We claim that we may assume that $C_S(r) \leq N_G(R^\dagger)$ and $|R \cap R^\dagger| \geq 2^2$. Let $T \in \mathrm{Syl}_2(C_G(r))$ such that $C_S(r) \leq T$. Since $R^\dagger \leq C_G(r)$, there exists $T_0 \in \mathrm{Syl}_2(C_G(r))$ such that $T_0 \geq R^\dagger$. Let $S^\dagger \in \mathrm{Syl}_2(L^\dagger)$ where $L^\dagger$ is G conjugate to L such that $T_0 \leq S^\dagger$. Hence $J(T_0) = R^\dagger$ by Lemmas 1.1.10 and 3.3.4. So $J(T)$ is conjugate to $J(T_0) = R^\dagger$ in $C_G(r)$. Therefore $C_S(r)$ normalizes a conjugate R^* of $R^\dagger$.

The largest elementary abelian subgroup of $N_G(R^*)/R^* \cong 3^3 : \mathrm{Sym}(4)^+$ has order 2^2. Hence $|C_R(r)R^*/R^*| \leq 2^2$. Since $C_R(r)R^*/R^* \cong C_R(r)/C_R(r) \cap R^*$ and by Lemma 3.2.4 (i), $|C_R(r)| = 2^4$, this implies that $|C_R(r) \cap R^*| \geq 2^2$. However, $R \cap R^* \geq C_R(r) \cap R^*$ and hence $|R \cap R^*| \geq 2^2$.

Note that $r \in R^*$ and so $R^* \cap S \nleq R$ and we may therefore replace $R^\dagger$ by R^* and satisfy $C_S(r) \leq N_G(R^\dagger)$ and $|R \cap R^\dagger| \geq 2^2$.

Suppose that $C_R(r) = R \cap R^\dagger$. Then by Lemma 3.2.4 (i), $|R \cap R^\dagger| = 2^4$ and $R \cap R^\dagger = [R, r]$. So,

$$\begin{aligned} [R, (R \cap R^\dagger)\langle r \rangle] &= [R, R \cap R^\dagger][R, \langle r \rangle] \\ &= [R, r] && \text{since } [R, R \cap R^\dagger] = 1 \\ &= R \cap R^\dagger \\ &< (R \cap R^\dagger)\langle r \rangle. \end{aligned}$$

Therefore, $R \leq N_G((R \cap R^\dagger)\langle r \rangle)$. Also, since $r \in R^\dagger$,

$$R^\dagger \leq C_G((R \cap R^\dagger)\langle r \rangle) \leq N_G((R \cap R^\dagger)\langle r \rangle).$$

Let $R \le T \in \mathrm{Syl}_2(N_G((R \cap R^\dagger)\langle r\rangle))$ and $R^\dagger \le T_0 \in \mathrm{Syl}_2(N_G((R \cap R^\dagger)\langle r\rangle))$. Since T and T_0 are conjugate in $N_G((R \cap R^\dagger)\langle r\rangle)$ and $R = J(T)$ and $R^\dagger = J(T_0)$, this implies that R and $R^\dagger$ are conjugate in $N_G((R \cap R^\dagger)\langle r\rangle)$. Therefore, $R \le C_G((R \cap R^\dagger)\langle r\rangle)$ and so $[R, r] = 1$. However, by Lemma 3.2.4 (i), $[R, r] = C_R(r)$ and $|C_R(r)| = 2^4$. Hence we have a contradiction and therefore $C_R(r) \ne R \cap R^\dagger$. In particular, $|R \cap R^\dagger| \in \{2^2, 2^3\}$.

Suppose $C_R(r) \le R^\dagger$. Then, by Lemma 3.2.4 (i), $2^4 = |C_R(r)| \le |R \cap R^\dagger| \in \{2^2, 2^3\}$, a contradiction. Hence $C_R(r) \not\le R^\dagger$.

Since $C_R(r) \not\le R^\dagger$, we have that $C_{R^\dagger}(C_R(r)) \ge \langle R \cap R^\dagger, r\rangle$. Therefore, as $r \notin R \cap R^\dagger$ and $|R \cap R^\dagger| \ge 2^2$, this implies that $|C_{R^\dagger}(C_R(r))| \ge 2^3$. Therefore, Lemma 3.2.4 (iii), implies that $|C_R(r)R^\dagger/R^\dagger| = 2$. Since $C_R(r)R^\dagger/R^\dagger \cong C_R(r)/(C_R(r) \cap R^\dagger)$ and $|C_R(r)| = 2^4$ by Lemma 3.2.4 (i), this implies that $|C_R(r) \cap R^\dagger| = 2^3$. Therefore, as $R \cap R^\dagger \ge C_R(r) \cap R^\dagger$ and $|R \cap R^\dagger| \ne 2^4$, we have that $|R \cap R^\dagger| = 2^3$. So $|C_{R^\dagger}(C_R(r))| \ge |R \cap R^\dagger||\langle r\rangle| = 2^4$. By Lemma 3.2.4 (i), $|C_{R^\dagger}(C_R(r))| \le 2^4$ and hence $|C_{R^\dagger}(C_R(r))| = 2^4$.

We claim $C_R(r)R^\dagger < C_S(r)R^\dagger$. Suppose $C_R(r)R^\dagger = C_S(r)R^\dagger$. Then $C_S(r) = C_S(r) \cap C_R(r)R^\dagger = C_R(r)(C_S(r) \cap R^\dagger)$, by Dedekind's modular law. Therefore, since $C_R(r)$ and $R^\dagger$ centralize $R \cap R^\dagger$, we have that $C_S(r)$ centralizes $R \cap R^\dagger$. Therefore, $2^3 = |R \cap R^\dagger| \le |C_R(C_S(r))| \in \{2, 2^2\}$ by (3.1), a contradiction and so $C_R(r)R^\dagger < C_S(r)R^\dagger$ as claimed.

Let $f \in C_S(r)$ such that $|\langle f, C_R(r)\rangle R^\dagger/R^\dagger| = 2^2$. Such an f exists since $|C_R(r)R^\dagger/R^\dagger| = 2$ and $C_R(r)R^\dagger < C_S(r)R^\dagger$. Since $\langle f, C_R(r)\rangle R^\dagger/R^\dagger \le 3^3 : \mathrm{Sym}(4)^+$, by Lemma 3.2.4 (ii), $|C_{R^\dagger}(\langle f, C_R(r)\rangle R^\dagger/R^\dagger)| = 2^2$. However, $C_{R^\dagger}(\langle f, C_R(r)\rangle R^\dagger/R^\dagger) = C_{C_{R^\dagger}(C_R(r))}(f)$ and so $|C_{C_{R^\dagger}(C_R(r))}(f)| = 2^2$. Since $|C_{R^\dagger}(C_R(r))| = 2^4$ and $R \cap R^\dagger \le C_{R^\dagger}(C_R(r))$ with $|R \cap R^\dagger| = 2^3$, Lemma 1.3.2 implies that $C_{C_{R^\dagger}(C_R(r))}(f) = C_{R \cap R^\dagger}(f)$. Therefore $C_{C_{R^\dagger}(C_R(r))}(f) \le R \cap R^\dagger$. However $r \in C_{C_{R^\dagger}(C_R(r))}(f)$ and $r \notin R \cap R^\dagger$. This is a contradiction and hence no such $R^\dagger$ and r can be chosen. Therefore, R is strongly closed in L. $\square$

The final result of this chapter will be applied to complete the proof of Theorem B in Chapter 6.

Theorem 3.3.6 *Suppose that G is a group, $R \le G$, $L = N_G(R)$ and $S \in \mathrm{Syl}_2(L)$. Assume that:*

(i) R is elementary abelian of order 2^8;

(ii) $R \le S$;

(iii) $L \ge L_0$ where $L_0/R \cong 3^3 : \mathrm{Sym}(4)^+$;

(iv) $\mathrm{Syl}_3(L_0) \subseteq \mathrm{Syl}_3(L)$;

(v) the elements in L_0/R-conjugacy class 3A act fixed-point-freely on R; and

(vi) L_0/R acts irreducibly on R.

Then either $G = O_{2'}(G)L$ or R is not strongly closed in S with respect to G.

Proof. Suppose that R is strongly closed in S with respect to G and set $M = \langle R^G \rangle$. We note that $R \trianglelefteq S$ since R is strongly closed in S. For $X \le G$, let $\overline{X} = X/O_{2'}(M)$. By Lemma 1.1.22, $\overline{R}$ is strongly closed in $\overline{S}$ with respect to $\overline{G}$. Since $O_2(\overline{M}) \le \overline{S}$ and $\overline{S} \in \mathrm{Syl}_2(\overline{L})$, we see that $O_2(\overline{M}) \le \overline{L}$. Suppose that $O_2(\overline{M}) > 1$. So $O_2(\overline{M}) \cap \overline{L} \ne 1$. Therefore, since $L_0 \le L$ acts irreducibly on R by assumption, $O_2(\overline{M}) \cap \overline{L} = \overline{R}$ and hence $\overline{R} = O_2(\overline{M})$. We have that $O_2(\overline{M})$ is a characteristic subgroup of $\overline{M} \triangleleft \overline{G}$ and so $N_{\overline{G}}(O_2(\overline{M})) = \overline{G}$. Since $\overline{R}$ is strongly closed in $\overline{S}$, we see that $N_{\overline{G}}(O_2(\overline{M}))$ normalizes R. Therefore $N_{\overline{G}}(O_2(\overline{M})) = N_{\overline{G}}(\overline{R}) = \overline{L}$ and thus $\overline{G} = \overline{L}$. Hence $G = O_{2'}(M)L = O_{2'}(G)L$ and we are done in this case.

So $O_2(\overline{M}) = 1$. Let J be such that $RJ \trianglelefteq L_0$ and $|J| = 3^3$ and assume without loss of generality that $\overline{S} \cap N_{\overline{L}_0}(\overline{J}) \in \mathrm{Syl}_2(N_{\overline{L}_0}(\overline{J}))$. So $(\overline{S} \cap N_{\overline{L}_0}(\overline{J}))\overline{R}/\overline{R} \in \mathrm{Syl}_2(\overline{L}_0/\overline{R})$ and therefore $(\overline{S} \cap N_{\overline{L}_0}(\overline{J}))\overline{R}/\overline{R} \cong \mathrm{Dih}(8)$ since $L_0/R \cong 3^3 : \mathrm{Sym}(4)^+$. We note that $\Omega_1(\overline{S}) = \overline{R}$

by Goldschmidt's Theorem, see Theorem 1.1.21. Since $\overline{RJ} \lhd \overline{L}_0$ and $\overline{J} \in \mathrm{Syl}_3(\overline{RJ})$, the Frattini Lemma implies that $N_{\overline{L}_0}(\overline{J})\overline{RJ} = N_{\overline{L}_0}(\overline{J})\overline{R} = \overline{L}_0$. We see that

$$[N_{\overline{L}_0}(\overline{J}) \cap \overline{R}, \overline{J}] \leq \overline{R} \cap \overline{J} = 1,$$

since $|R|$ and $|J|$ are coprime. Therefore $N_{\overline{L}_0}(\overline{J}) \cap \overline{R} \leq C_{\overline{R}}(\overline{J})$. Since J contains elements in L_0-conjugacy class 3A and these act fixed-point-freely on R, this implies that $N_{\overline{L}_0}(\overline{J}) \cap \overline{R} = 1$. Hence $N_{\overline{L}_0}(\overline{J})$ is a complement to $\overline{R}$ in $\overline{L}_0$ and $\overline{S} \cap N_{\overline{L}_0}(\overline{J}) \cong \mathrm{Dih}(8)$. Therefore $\overline{R} = \Omega_1(\overline{S}) \geq \overline{S} \cap N_{\overline{L}_0}(\overline{J})$ which contradicts the fact that $N_{\overline{L}_0}(\overline{J}) \cap \overline{R} = 1$.

Therefore, if R is strongly closed in S with respect to G, $G = O_{2'}(G)L$, otherwise R is not strongly closed in S and hence the result holds. $\square$

Chapter 4

The Structure of Amalgams of Types $\mathrm{G}_2(3)$ and F_3

In this chapter we establish our first results concerning the subgroup structure and other properties of amalgams of types $\mathrm{G}_2(3)$ and F_3.

4.1 Properties of Amalgams of Type $\mathrm{G}_2(3)$

We consider amalgams of type $\mathrm{G}_2(3)$. Throughout this section we let $\mathcal{G} = \mathcal{G}(G_\alpha, G_\beta, G_{\alpha\beta})$ be an amalgam of type $\mathrm{G}_2(3)$, as defined in Definition 1.6.2. We use the notation established in Sections 1.4, 1.5 and 1.6.

Lemma 4.1.1 *The following hold in the amalgam $\mathcal{G}$.*

(i) $S_{\alpha\beta} = Q_\alpha Q_\beta$.

(ii) $Z_\alpha \cap Z_\beta = Z(L_\alpha)Z(L_\beta) = Z(S_{\alpha\beta})$.

(iii) $|Z_\alpha \cap Z_\beta| = 3^2$.

(iv) *For* $\{\gamma, \delta\} = \{\alpha, \beta\}$, $Z_\gamma \leq Q_\delta$.

(v) $Z_\alpha Z_\beta = Q_\alpha \cap Q_\beta$.

(vi) $|Q_\alpha \cap Q_\beta| = 3^4$.

Proof. (i) We have that $Q_\alpha \neq Q_\beta$ as $\mathcal{G}$ is a simple amalgam and $|S_{\alpha\beta} : Q_\alpha| = |S_{\alpha\beta} : Q_\beta| = 3$ by Definition 1.6.2. Hence $Q_\alpha Q_\beta = S_{\alpha\beta}$.

(ii) Definition 1.6.2 implies that $Z(L_\alpha) \cong Z(L_\beta)$ has order 3. Let $\{\gamma, \delta\} = \{\alpha, \beta\}$. By (i), we have that $Q_\gamma \leq L_\delta$ and hence $[Z(L_\delta), Q_\gamma] = 1$. Since $C_{L_\gamma}(Q_\gamma) \leq Q_\gamma$, for $\gamma \in \{\alpha, \beta\}$, we have that $Z(L_\alpha)Z(L_\beta) \leq Q_\alpha \cap Q_\beta$. This is centralized by $Q_\alpha Q_\beta = S_{\alpha\beta}$. In particular, $Z(L_\alpha)Z(L_\beta) \leq Z_\alpha \cap Z_\beta$. Therefore, since $Z_\alpha \neq Z_\beta$ and $|Z_\gamma| = 3^3$, we have that

$$Z_\alpha \cap Z_\beta = Z(L_\alpha)Z(L_\beta) = Z(S_{\alpha\beta}).$$

(iii) This follows immediately from (ii).

(iv) Let $\{\gamma, \delta\} = \{\alpha, \beta\}$ and suppose that $Z_\gamma \not\leq Q_\delta$. Then

$$Z_\gamma > Z_\gamma \cap Q_\delta \geq Z_\alpha \cap Z_\beta = Z(S_{\alpha\beta}),$$

by (ii). Hence $Z_\gamma \cap Q_\delta = Z(S_{\alpha\beta})$, by orders. So,

$$[Z_\gamma, Q_\delta] \leq Z_\gamma \cap Q_\delta = Z(S_{\alpha\beta}) \leq Z_\delta.$$

Therefore, Z_γ centralizes the non-central chief factor Q_δ/Z_δ. This is a contradiction and hence $Z_\gamma \leq Q_\delta$.

(v) Let $\{\gamma, \delta\} = \{\alpha, \beta\}$. We have that $Q_\alpha \neq Q_\beta$, by Definition 1.6.2 and hence $|Q_\gamma : Q_\alpha \cap Q_\beta| \geq 3$. Since $Z_\alpha \neq Z_\beta$ and by (iv), $Z_\gamma \leq Q_\delta$, $|Q_\alpha \cap Q_\beta : Z_\gamma| \geq 3$. Thus, as $|Q_\gamma : Z_\gamma| = 3$, we see that $|Q_\alpha \cap Q_\beta : Z_\gamma| = 3$. Therefore $Z_\alpha Z_\beta = Q_\alpha \cap Q_\beta$.

(vi) This follows immediately from (v). $\square$

Lemma 4.1.2 *(i) If $A \leq S_{\alpha\beta}$ is elementary abelian, then $|A| \leq 3^4$.*

(ii) $Z_\alpha Z_\beta$ is elementary abelian, of order 3^4.

(iii) There exists $A \leq S_{\alpha\beta}$ with $A \neq Z_\alpha Z_\beta$ such that A is elementary abelian of order 3^4.

Proof. (i) Since $Z(S_{\alpha\beta}) \neq S_{\alpha\beta}$, certainly $S_{\alpha\beta}$ is not elementary abelian. Let $B \leq S_{\alpha\beta}$ be elementary abelian with $|B| = 3^5$. Suppose that $Z_\alpha \leq B$. Then $B \leq C_{S_{\alpha\beta}}(Z_\alpha) = Q_\alpha$. Hence, by orders, $B = Q_\alpha$, and Q_α is elementary abelian, a contradiction. Therefore, as $Q_\alpha = C_{S_{\alpha\beta}}(Z_\alpha)$, $Z_\alpha \nleq B$. So $|B \cap Q_\alpha| = 3^4$ and hence $Q_\alpha = Z_\alpha(B \cap Q_\alpha)$. So Q_α is elementary abelian, again a contradiction. Therefore, if $A \leq S_{\alpha\beta}$ is elementary abelian, then $|A| \leq 3^4$.

(ii) By Lemma 4.1.1 (iv), $Z_\alpha \leq Q_\beta$. Hence $[Z_\alpha, Z_\beta] = 1$ and therefore $Z_\alpha Z_\beta$ is elementary abelian and has order 3^4, by Lemma 4.1.1 (v) and (vi).

(iii) Suppose $Z_\alpha Z_\beta \trianglelefteq G_\alpha$. We have that Q_α/Z_α is a natural G_α/Q_α-module and hence Q_α/Z_α is a minimal normal subgroup of G_α/Z_α. However, this gives rise to a contradiction as $Z_\alpha Z_\beta < Q_\alpha$. Hence $Z_\alpha Z_\beta \ntrianglelefteq G_\alpha$.

Choose $x \in G_\alpha \backslash G_{\alpha\beta}$ such that $A = (Z_\alpha Z_\beta)^x \neq Z_\alpha Z_\beta$. Such a choice is possible since $Z_\alpha Z_\beta \ntrianglelefteq G_\alpha$. However, A is elementary abelian of order 3^4 since it is conjugate to $Z_\alpha Z_\beta$ and $A \leq Q_\alpha^x = Q_\alpha \leq S_{\alpha\beta}$. $\square$

Lemma 4.1.3 *The amalgam $\mathcal{G}$ has the following properties.*

(i) Q_γ has exponent 3, for $\gamma \in \{\alpha, \beta\}$.

(ii) If $x \in Q_\alpha \backslash Q_\beta$ and $y \in Q_\beta \backslash Q_\alpha$, then $[y, x, x] \neq 1 \neq [x, y, y]$.

(iii) *If $z \in S_{\alpha\beta}$ has order 3, then $z \in Q_\alpha \cup Q_\beta$.*

(iv) *Let G be a faithful completion of $\mathcal{G}$. Then $N_{N_G(Q_\gamma)}(S_{\alpha\beta}) \leq N_G(Q_\delta)$, where $\{\gamma, \delta\} = \{\alpha, \beta\}$.*

Proof. (i) [32, Lemma 6.4, (ii)] By Lemma 4.1.1 (v), $Z_\alpha Z_\beta = Q_\alpha \cap Q_\beta$. Let $\gamma \in \{\alpha, \beta\}$. Since Z_γ is elementary abelian by definition, the elements of the cosets of Z_γ in Q_γ that also lie in $Z_\alpha Z_\beta$ have order dividing 3. We have that $L_\gamma/Q_\gamma \cong \mathrm{SL}_2(3)$ acts transitively on the non-trivial elements of Q_γ/Z_γ and therefore, the elements of every coset of Z_γ in Q_γ have order dividing 3. Hence Q_γ has exponent 3.

(ii) [32, Lemma 6.4, (iii)] Let $x \in Q_\alpha \backslash Q_\beta$ and $y \in Q_\beta \backslash Q_\alpha$ We show that $[y, x, x] \neq 1$, the proof for $[x, y, y]$ is similar. Since Q_α/Z_α is a L_α/Q_α-module, $y \notin Q_\alpha$ and $x \notin Q_\alpha \cap Q_\beta$, we have $[y, x] \notin Z_\alpha$. Suppose $[y, x, x] = 1$. Then $C_{Q_\alpha \cap Q_\beta}(x) \geq \langle Z_\alpha, [x, y]\rangle$ and $|\langle Z_\alpha, [x, y]\rangle| = 3^4$. Since $Q_\alpha \cap Q_\beta = Z_\alpha Z_\beta$ by Lemma 4.1.1, $C_{Z_\beta}(x) \geq Z_\beta \cap \langle Z_\alpha, [x, y]\rangle$ and therefore $|C_{Z_\beta}(x)| = 3^3$. However, $Z_\beta/Z(L_\beta)$ is a natural L_β/Q_β-module and $x \notin Q_\beta$. Hence $|C_{Z_\beta}(x)| \leq 3^2$, giving a contradiction and therefore $[y, x, x] \neq 1$.

(iii) [32, Lemma 6.4, (iv)] Let $z \in S_{\alpha\beta} \backslash (Q_\alpha \cup Q_\beta)$. So $z = xy$ for $x \in Q_\alpha \backslash Q_\beta$ and $y \in Q_\beta \backslash Q_\alpha$ by Lemma 4.1.1 (i). Both x and y have order 3 by (i) and hence z has order 3 or 3^2. Suppose z has order 3. We also have that commutators of the form $[y, x, x]$ and $[y, x, y]$ are central in $S_{\alpha\beta}$. So

$$\begin{aligned}
1 &= xyxyxy \\
&= x^2y[y,x]yxy \\
&= x^2y^2[y,x][y,x,y]xy \\
&= x^2y^2x[y,x]y[y,x,x][y,x,y] \\
&= x^2y^2xy[y,x][y,x,x][y,x,y]^2
\end{aligned}$$

$$
\begin{aligned}
&= y^2[y^2,x]y[y,x][y,x,x][y,x,y]^2 \\
&= [y^2,x][y,x][y^2,x,y][y,x,x][y,x,y]^2 \\
&= [y^3,x][y^2,x,y]^{-1}[y^2,x,y][y,x,x][y,x,y^2] \\
&= [y,x,x][y,x,y]^2.
\end{aligned}
$$

So $[y,x,x] = [y,x,y]$. However $[y,x,x] \in Z(L_\alpha)$, $[y,x,y] \in Z(L_\beta)$ and $Z(L_\alpha) \cap Z(L_\beta) = 1$. So $[y,x,x] = [y,x,y] = 1$ and this contradicts (ii). Hence z has order 3^2.

(iv) [32, Lemma 6.4, (v)] Since G is a faithful completion of $\mathcal{G}$, we have that $N_G(S_{\alpha\beta})$ conjugates elements of $S_{\alpha\beta}$ of order 3 to elements of order 3. Hence, by (iii), $N_G(S_{\alpha\beta})$ preserves the set $\{Q_\alpha, Q_\beta\}$. Therefore the lemma holds. $\square$

Figure 4.1 indicates the inclusions among subgroups in the amalgam $\mathcal{G}$, including the results proven in this section.

4.2 Properties of Amalgams of Type F_3

We now consider amalgams of type F_3. We let $\mathcal{F}_3 = \mathcal{F}_3(G_\alpha, G_\beta, G_{\alpha\beta})$ be an amalgam of type F_3, as defined in Definition 1.6.3 throughout this section. As in the previous section, we use the notation established in Sections 1.4, 1.5 and 1.6. We prove a number of results about the structure of $\mathcal{F}_3$.

Lemma 4.2.1 *The following hold in the amalgam $\mathcal{F}_3$.*

(i) $S_{\alpha\beta} = Q_\alpha Q_\beta$.

(ii) $Z_\alpha \leq [Q_\alpha, Q_\alpha]$.

(iii) $C_{L_\alpha}(Z_\alpha) = C_{S_{\alpha\beta}}(Z_\alpha) = Q_\alpha$.

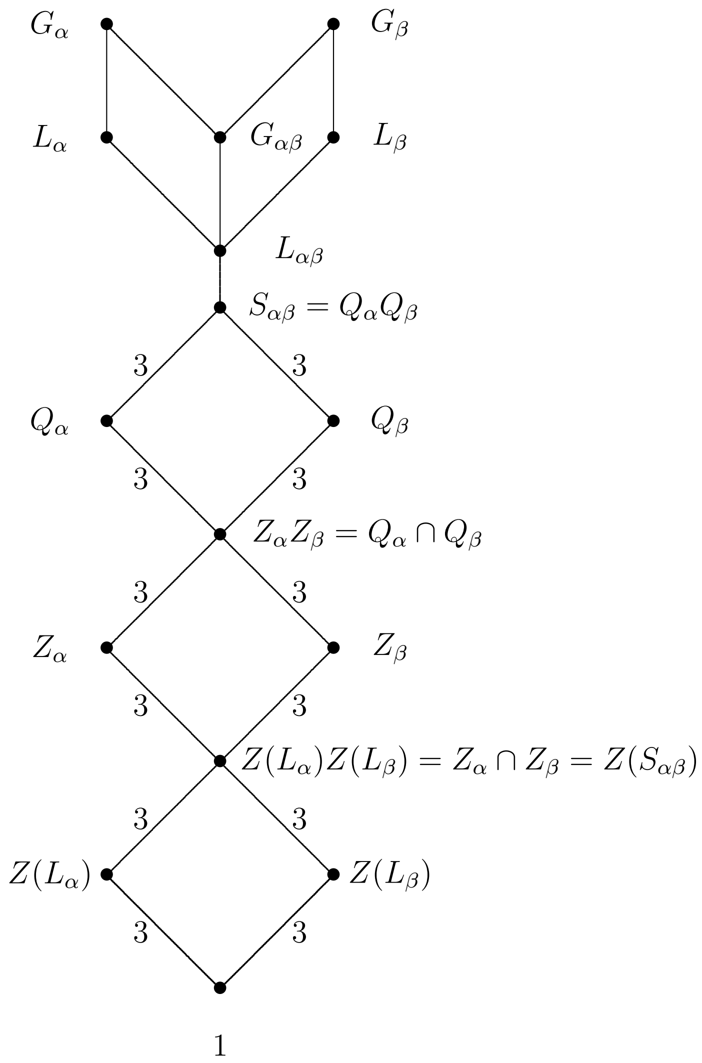

Figure 4.1: Structure of $\mathcal{G}$

(iv) $Z_\beta = Z(S_{\alpha\beta})$.

(v) $Z_\alpha \leq Z_2(S_{\alpha\beta})$.

(vi) $Z(C_\beta) = C_{Q_\beta}(C_\beta)$.

(vii) $Z(C_\beta) = V_\beta$.

Proof. (i) Since $\mathcal{F}_3$ is a simple amalgam and $|S_{\alpha\beta} : Q_\alpha| = |S_{\alpha\beta} : Q_\beta| = 3$ by Definition 1.6.2, we have that $Q_\alpha \neq Q_\beta$. Hence $Q_\alpha Q_\beta = S_{\alpha\beta}$.

(*ii*) Certainly, $[Q_\alpha, Q_\alpha] \trianglelefteq Q_\alpha$. If $[Q_\alpha, Q_\alpha] = 1$, then Q_α is elementary abelian and $Z(Q_\alpha) = Q_\alpha$. Since, $\Omega_1(Z(Q_\alpha)) = Z_\alpha$ has order 3^2, this is a contradiction. Hence $[Q_\alpha, Q_\alpha] \neq 1$. Therefore, by Lemma 1.1.12, $[Q_\alpha, Q_\alpha] \cap Z(Q_\alpha) \neq 1$. This implies that $[Q_\alpha, Q_\alpha] \cap \Omega_1(Z(Q_\alpha)) = [Q_\alpha, Q_\alpha] \cap Z_\alpha \neq 1$. Since $[Q_\alpha, Q_\alpha]$ is a characteristic subgroup of $Q_\alpha \triangleleft L_\alpha$, we have that $[Q_\alpha, Q_\alpha] \triangleleft L_\alpha$. Hence, as Z_α is a minimal normal subgroup of L_α, we have that $Z_\alpha \leq [Q_\alpha, Q_\alpha]$.

(*iii*) We have that $Q_\alpha \leq C_{L_\alpha}(Z_\alpha) \trianglelefteq L_\alpha$ and hence $Q_\alpha \leq C_{S_{\alpha\beta}}(Z_\alpha) \leq S_{\alpha\beta}$. Since $|S_{\alpha\beta} : Q_\alpha| = 3$, this implies that either $C_{S_{\alpha\beta}}(Z_\alpha) = S_{\alpha\beta}$, or $C_{S_{\alpha\beta}}(Z_\alpha) = Q_\alpha$. Suppose the former holds. Then $C_{L_\alpha}(Z_\alpha) \neq Q_\alpha$, since $C_{S_{\alpha\beta}}(Z_\alpha) \leq C_{L_\alpha}(Z_\alpha)$. Therefore, as $C_{L_\alpha}(Q_\alpha) \leq C_{L_\alpha}(Z_\alpha)$ and $C_{L_\alpha}(Q_\alpha) = L_\alpha$, we have that $C_{L_\alpha}(Z_\alpha) = L_\alpha$. However, since $Z_\beta \leq Z_\alpha$, this implies that $Z_\beta \trianglelefteq L_\alpha$, which contradicts the simplicity of $\mathcal{F}_3$. Hence $C_{S_{\alpha\beta}}(Z_\alpha) = Q_\alpha$ as required.

(*iv*) We have that $Z(S_{\alpha\beta}) \leq Q_\beta$, otherwise Q_β would not contain any non-central chief factors for L_β/Q_β, a contradiction. Hence $Z(S_{\alpha\beta}) \leq C_{Q_\beta}(Q_\beta) = Z_\beta$. Since $|Z_\beta| = 3$ and $Z(S_{\alpha\beta}) \neq 1$ as $S_{\alpha\beta}$ is a 3-group, we have that $Z(S_{\alpha\beta}) = Z_\beta$.

(*v*) Since $Z_\alpha/Z_\beta \trianglelefteq S_{\alpha\beta}/Z_\beta$ and $|Z_\alpha/Z_\beta| = 3$, we have that $Z_\alpha/Z_\beta \leq Z(S_{\alpha\beta}/Z_\beta)$ by Corollary 1.1.14. Hence, as $Z_\beta = Z(S_{\alpha\beta})$, by (*iv*), $Z_\alpha \leq Z_2(S_{\alpha\beta})$.

(*vi*) Since $V_\beta \leq C_\beta \leq Q_\beta$, we have that $C_{Q_\beta}(C_\beta) \leq C_{Q_\beta}(V_\beta) \leq C_{L_\beta}(V_\beta) = C_\beta$. Hence $C_{Q_\beta}(C_\beta) = Z(C_\beta)$ as required.

(*vii*) By Lemma (*vi*), $Z(C_\beta) = C_{Q_\beta}(C_\beta)$. Since $C_\beta = C_{L_\beta}(V_\beta)$, we see that $V_\beta \leq C_{Q_\beta}(C_\beta)$. Suppose that $V_\beta < Z(C_\beta)$. We have that $U_\alpha \leq W_\beta \leq C_\beta \leq Q_\alpha$ and hence $Z(C_\beta) = C_{Q_\beta}(C_\beta) \leq C_{Q_\beta}(U_\alpha) = U_\alpha$, by Lemma 4.2.2. Since $[U_\alpha : Z(W_\beta)] = 3$ and $U_\alpha \neq Z(C_\beta)Z(W_\beta)$, we see that $Z(C_\beta) \leq Z(W_\beta)$ otherwise $Z(C_\beta) = U_\alpha$. Hence $Z(C_\beta) = Z(W_\beta)$ as $[Z(W_\beta), V_\beta] = 3$ and $Z(C_\beta) > V_\beta$ by assumption. Let

$g \in L_\alpha \backslash N_{L_\alpha}(S_{\alpha\beta})$ and define $Q_{\alpha-1} = Q_\beta^g$, $C_{\alpha-1} = C_\beta^g$ and $Z(W_{\alpha-1}) = Z(W_\beta)^g$. Then $C_\beta \cap C_{\alpha-1}$ centralizes $Z(W_{\alpha-1})Z(W_\beta)$. Since $Z(W_{\alpha-1}) \neq Z(W_\beta)$, we have that $Z(W_{\alpha-1})Z(W_\beta) = U_\alpha$. Hence $C_\beta \cap C_{\alpha-1} = U_\alpha$ since $C_{Q_\alpha}(U_\alpha) = U_\alpha$ by Lemma 4.2.2. As $|C_\beta/U_\alpha| = 3^2$ and $|Q_\alpha/U_\alpha| = 3^4$, we have that $Q_\alpha = C_\beta C_{\alpha-1}$. Therefore $Z(W_\beta) \cap Z(W_{\alpha-1})$ is centralized by Q_α. Since $|Z(W_\beta)| = 3^4$ and $|U_\alpha| = 3^5$, we see that $Z(W_\beta) \cap Z(W_{\alpha-1}) > Z_\beta$ which is a contradiction. Therefore $C_{Q_\beta}(C_\beta) = Z(C_\beta) = V_\beta$. $\square$

Lemma 4.2.2 *[32, Lemma 13.1] The following holds for the amalgam $\mathcal{F}_3$.*

(i) U_α is elementary abelian and

$$\Phi(Q_\alpha) = [Q_\alpha, Q_\alpha] = U_\alpha = C_{Q_\alpha}(U_\alpha).$$

(ii) $[W_\beta, W_\beta] = \Phi(W_\beta) = Z_\beta$.

Proof. Since Z_α is elementary abelian and $V_\beta \leq Z(W_\beta)$ we have that $V_\beta = \langle Z_\alpha^{L_\beta} \rangle$ is elementary abelian. Therefore, as $U_\alpha \leq C_\beta \cap Q_\alpha = C_{Q_\alpha}(V_\beta)$ we see that $U_\alpha = \langle V_\beta^{L_\alpha} \rangle$ is also elementary abelian. Since Q_α/U_α is a L_α/Q_α-module, it is elementary abelian. Also, $Q_\alpha/\Phi(Q_\alpha)$ is elementary abelian and hence,

$$U_\alpha \geq \Phi(Q_\alpha) \geq [Q_\alpha, Q_\alpha]. \tag{4.1}$$

By Lemma 4.2.1 (*ii*), $Z_\alpha \leq [Q_\alpha, Q_\alpha]$. Since U_α/Z_α is an irreducible $\Omega_3(3)$-module, either $[Q_\alpha, Q_\alpha] = Z_\alpha$, or $[Q_\alpha, Q_\alpha] = U_\alpha$. Suppose that the former holds. Since

$$[Q_\alpha \cap Q_\beta, Q_\alpha] \leq [Q_\alpha, Q_\alpha] = Z_\alpha \leq V_\beta,$$

we have that $[(Q_\alpha \cap Q_\beta)/V_\beta, Q_\alpha] = 1$. Therefore, $(Q_\alpha \cap Q_\beta)/V_\beta \leq C_{(Q_\alpha \cap Q_\beta)/V_\beta}(Q_\alpha) \leq C_{Q_\beta/V_\beta}(Q_\alpha) \leq Q_\beta/V_\beta$. Since $|Q_\beta/(Q_\alpha \cap Q_\beta)| = 3$, we have that $[Q_\beta/V_\beta : C_{Q_\beta/V_\beta}(Q_\alpha)] \leq 3$ and hence Q_β/V_β has at most one non-central chief factor for L_β/Q_β. This is a contradiction since Q_β/C_β and $W_\beta/Z(W_\beta)$ are both non-central chief factors for L_β/Q_β by definition. Hence $U_\alpha = [Q_\alpha, Q_\alpha]$ and equality holds in Equation 4.1.

Let $F = C_{Q_\alpha}(U_\alpha)$. Then $U_\alpha \leq F$ as U_α is elementary abelian. So suppose that $U_\alpha < F$. Since the L_α-chief factors of Q_α/U_α are both $\mathrm{SL}_2(3)$-modules, we have that either $[Q_\alpha : F] = 3^2$ or $F_\alpha = Q_\alpha$. We have that $V_\beta \leq U_\alpha$. Therefore $F \leq C_{Q_\alpha}(V_\beta)$. Since V_β contains V_β/Z_β, a non-central L_β-chief factor, we have that $C_{Q_\alpha}(V_\beta) \leq C_{L_\beta}(V_\beta) = C_\beta$. Therefore $F \leq C_\beta \leq Q_\beta$. Hence $F \neq Q_\alpha$ and $[Q_\alpha : F] = 3^2$. So $|F| = |C_\beta|$ and hence they are equal. Therefore $F \trianglelefteq \langle L_\alpha, L_\beta \rangle$ which contradicts the simplicity of $\mathcal{F}_3$. Hence $F = C_{Q_\alpha}(U_\alpha) = U_\alpha$ and (i) holds.

Since U_α is elementary abelian by (i), and W_β centralizes $Z(W_\beta)$, we have that

$$Z(W_\beta) \leq C_{Q_\beta}(W_\beta) \leq C_{Q_\alpha}(U_\alpha) = U_\alpha.$$

Hence, by orders $[U_\alpha : Z(W_\beta)] = 3$. We also have that $[W_\beta : U_\alpha] = 3$. Let x be the automorphism of order 3 that W_β/U_α induces on U_α. So $C_{U_\alpha}(x) = C_{U_\alpha}(W_\beta) = Z(W_\beta)$ and $[U_\alpha, x] = [U_\alpha, W_\beta]$. We have that $|U_\alpha/Z(W_\beta)| = 3$, so $\dim U_\alpha/C_{U_\alpha}(x) = 1$. Hence, by Lemma 1.3.1 (i), $\dim[U_\alpha, x] = 1$ and therefore $|[U_\alpha, W_\beta]| = 3$. Since $[U_\alpha, W_\beta] \triangleleft G_\beta$ and $Z_\beta = \Omega_1(Z(G_\beta))$, we have that $[U_\alpha, W_\beta] = Z_\beta$. We also note that $Z_\beta \trianglelefteq L_\beta$. So

$$[W_\beta, W_\beta] = \langle [U_\alpha, W_\beta]^{L_\beta} \rangle = \langle Z_\beta^{L_\beta} \rangle = Z_\beta.$$

Since $W_\beta = \langle U_\alpha^{L_\beta} \rangle$ and U_α is elementary abelian, we have that W_β, and hence W_β/Z_β is generated by elements of order 3. Therefore, as $W_\beta/Z_\beta = W_\beta/[W_\beta, W_\beta]$ is abelian, it is elementary abelian. Hence

$$\Phi(W_\beta) \leq Z_\beta = [W_\beta, W_\beta] \leq \Phi(W_\beta),$$

and equality holds, completing the proof of the lemma. $\square$

Figure 4.2 indicates the inclusions among subgroups in the amalgam $\mathcal{F}_3$, including the results proven in this section.

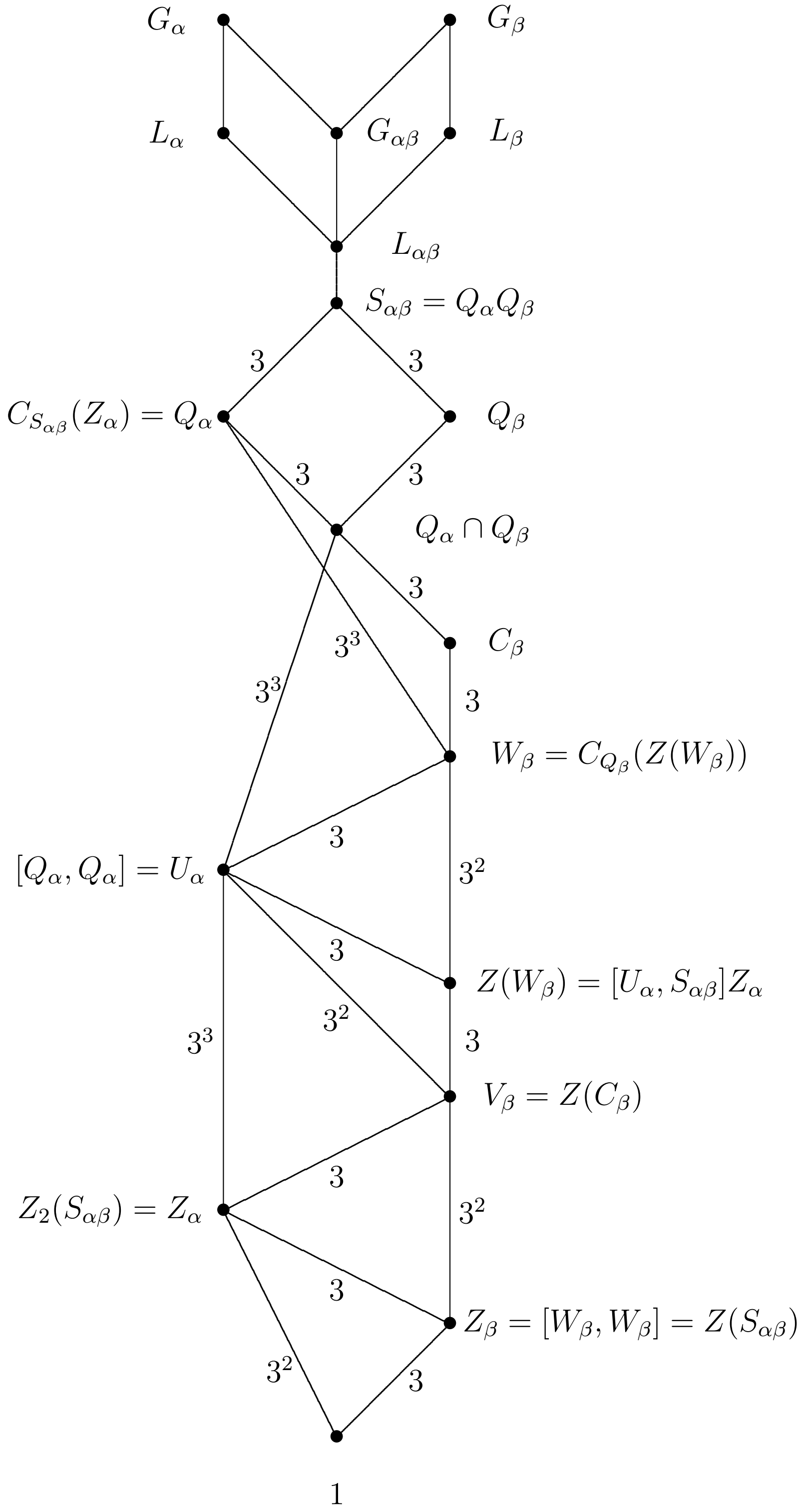

Figure 4.2: Structure of $\mathcal{F}_3$

Chapter 5

Proof of Theorem A

In this chapter we first prove a number of results concerning the coset graph of an amalgam of type F_3. We then prove Theorem A, using in particular the results from Section 1.7 and Chapter 2.

5.1 The Coset Graph of an Amalgam of Type F_3

Let $\mathcal{F}_3 = \mathcal{F}_3(G_\alpha, G_\beta, G_{\alpha\beta})$ be the amalgam of type F_3 from Section 4.2, G be a faithful completion of this amalgam such that $N_G(Z(L_\beta)) = G_\beta$ and $\Gamma = \Gamma(G, G_\alpha, G_\beta, G_{\alpha\beta})$ be the right coset graph. Let Γ_α and Γ_β denote the vertices of Γ in the α and β orbits respectively. Suppose $S_{\alpha\beta} \in \mathrm{Syl}_3(G_{\alpha\beta})$ and let T be a complement to $S_{\alpha\beta}$ in $G_{\alpha\beta}$. Let Θ be the connected subgraph of Γ that is fixed by T and contains the edge $\{\alpha, \beta\}$ and $\Theta_\beta = \Theta \cap \Gamma_\beta$.

Lemma 5.1.1 *The following hold.*

(i) The critical distance $b = \min_{\theta,\rho\in\Gamma}\{d(\theta,\rho) \mid Z_\theta \not\leq Q_\rho\}$ *is* 5.

(ii) G *acts locally* 7*-arc transitively on* Γ.

(iii) For $\theta \in \Theta$, *there exists a unique element* $t_\theta \in T^\#$ *such that* $t_\theta Q_\theta \in Z(L_\theta/Q_\theta)$.

(iv) *Let* $(\theta, \theta+1, \theta+2, \theta+3)$ *be a path of length* 3 *in* Θ*, with* $\theta \in \Theta_\beta$*. Then* $t_\theta = t_{\theta+3}$ *and* $T^\# = \{t_\theta, t_{\theta+1}, t_{\theta+2}\}$*.*

(v) T *is elementary abelian of order* 4*.*

(vi) *The elements of* T *are* G*-conjugate.*

(vii) Γ *has valency* 4*.*

Proof. (i) See [9, page 98].

(ii) See [9, (3.4), pages 74 and 98].

(iii) This follows since $L_\theta/Q_\theta \cong \mathrm{SL}_2(3)$ and $Z(\mathrm{SL}_2(3))$ has order 2.

(iv) See [9, (6.9)].

(v) Clearly $|T| = 4$ by part (iv). By part (iii), all non-trivial elements of T are involutions.

(vi) We consider the path $(\theta, \theta+1, \theta+2, \theta+3)$ of length 3 in Θ. By (iv), $T^\# = \{t_\theta, t_{\theta+1}, t_{\theta+2}\}$. Now, t_θ and $t_{\theta+2}$ are G-conjugate since G_θ and $G_{\theta+2}$ are G-conjugate by Lemma 1.5.3 (iii). Similarly $t_{\theta+1}$ and $t_{\theta+3}$ are G-conjugate. However, again by (iv), $t_{\theta+3} = t_\theta$ and hence the result follows.

(vii) Since T is a complement to $S_{\alpha\beta}$ in $G_{\alpha\beta}$ and T has order 4 by part (v), we see that $|G_{\alpha\beta} : S_{\alpha\beta}| = 4$. Let $\gamma \in \{\alpha, \beta\}$. We have that $|G_\gamma : Q_\gamma| = |\mathrm{GL}_2(3)| = 2^4 3$ by definition and $|S_{\alpha\beta} : Q_\gamma| = 3$. Hence

$$|G_\gamma : G_{\alpha\beta}| = \frac{|G_\gamma : Q_\gamma|}{|G_{\alpha\beta} : S_{\alpha\beta}||S_{\alpha\beta} : Q_\gamma|} = \frac{2^4 3}{2^2 3} = 2^2.$$

So $|\Gamma(\gamma)| = 4$ by Lemma 1.5.4, (i). Hence Γ has valency 4. □

Remark 5.1.2 We note that given any vertex $\gamma \in \Gamma$, we can find a conjugate $T^\dagger$ of T which fixes a path on which γ lies such that $T^\dagger$ is a complement to a Sylow 3-subgroup of a group conjugate to $G_{\alpha\beta}$. Therefore, by Lemma 5.1.1 (*iii*), we can find a unique non-trivial involution t_γ in a conjugate of T associated with γ such that $t_\gamma Q_\gamma \in Z(L_\gamma/Q_\gamma)$. For the rest of this section, for $\gamma \in \Gamma$, we let t_γ be this involution.

We require the following lemma concerning the group generated by two Sylow 3-subgroups of $\mathrm{SL}_2(3)$ in order to prove Lemma 5.1.4.

Lemma 5.1.3 *Let $P_1, P_2 \in \mathrm{Syl}_3(\mathrm{SL}_2(3))$ such that $P_1 \neq P_2$. Then $\langle P_1, P_2\rangle = \mathrm{SL}_2(3)$.*

Proof. Since $|\mathrm{SL}_2(3)| = 2^3 3$, we see that $\mathrm{SL}_2(3)$ contains four Sylow 3-subgroups. Since $P_1 \neq P_2$, $\langle P_1, P_2\rangle$ must contain four Sylow 3-subgroups of $\mathrm{SL}_2(3)$. Therefore $\langle P_1, P_2\rangle = \langle P \mid P \in \mathrm{Syl}_3(\mathrm{SL}_2(3))\rangle = \mathrm{SL}_2(3)$. □

Lemma 5.1.4 *Let $\gamma \in \Gamma$. Then G_γ induces $\mathrm{Sym}(4)$ on $\Gamma(\gamma)$. The kernel of this action is $Q_\gamma\langle t_\gamma\rangle$.*

Proof. We have that G_γ acts on $\Gamma(\gamma)$ in the same way that G_γ acts on the four cosets of $G_{\gamma\delta}$, where $\delta \in \Gamma(\gamma)$, in G_γ. So we consider the action of G_γ/Q_γ on $G_{\gamma\delta}/Q_\gamma$. By Lemma 5.1.1 (*vii*), $|G_\gamma : G_{\gamma\delta}| = 4$. Therefore, since $G_\gamma/Q_\gamma \cong \mathrm{GL}_2(3)$ it suffices to consider the action of $\mathrm{GL}_2(3)$ on the cosets of a subgroup of $\mathrm{GL}_2(3)$ of index four. Let $H = \mathrm{GL}_2(3)$ and $B = \left\{ \left(\begin{array}{cc} \lambda & 0 \\ a & \mu \end{array} \right) \middle| \lambda, \mu \in \mathrm{GF}(3)^*, a \in \mathrm{GF}(3) \right\}$. Then $|H : B| = 4$ since $|B| = 2^2 3$. We show that subgroups isomorphic to B are unique up to conjugacy. Certainly, $B \geq P$, where $P \in \mathrm{Syl}_3(H)$. So we have $|H : B| = 4$ and $|B : P| = 4$. Suppose that $P \ntrianglelefteq B$. Then there exists $P_1 \leq B$ such that $P_1 \in \mathrm{Syl}_3(H)$. Therefore by Lemma 5.1.3, $B \geq \mathrm{SL}_2(3)$. This is a contradiction since $|H : B| = 4$. Hence B normalizes P and therefore B is unique up to conjugacy.

Let ϕ be the action of $\mathrm{GL}_2(3)$ on the right cosets of B. Then $\phi : \mathrm{GL}_2(3) \to \mathrm{Sym}(4)$. We have that $\ker\phi = \bigcap_{g\in H} B^g$. In other words, $\ker\phi$ is the largest normal subgroup of H that is in B. Let $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$B^x = \left\{ \begin{pmatrix} \mu & a \\ 0 & \lambda \end{pmatrix} \middle| \lambda, \mu \in \mathrm{GF}(3)^*, a \in \mathrm{GF}(3) \right\}.$$

Hence,

$$\ker\phi \subseteq B \cap B^x = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \middle| \lambda, \mu \in \mathrm{GF}(3)^* \right\}.$$

Let $\lambda, \mu \in \mathrm{GF}(3)^*$ and $a \in \mathrm{GF}(3)$. Since

$$\begin{pmatrix} \lambda & 0 \\ a & \mu \end{pmatrix}^{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} = \begin{pmatrix} \lambda & 0 \\ -a & \mu \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ a & \mu \end{pmatrix}^{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}},$$

we see that $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \notin \ker\phi$. Therefore,

$$\ker\phi = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \cong Z(\mathrm{GL}_2(3)).$$

Let $X = G_\gamma/Q_\gamma$. Then by the First Isomorphism Theorem,

$$X/\ker\phi \cong \mathrm{GL}_2(3)/\ker\phi \cong \mathrm{PGL}_2(3) \cong \mathrm{Sym}(4).$$

Hence G_γ/Q_γ acts as $\mathrm{Sym}(4)$ on the cosets of $G_{\gamma\delta}/Q_\gamma$ and thus G_γ induces $\mathrm{Sym}(4)$ on

$\Gamma(\gamma)$.

Let ψ be the action of G_γ on $\Gamma(\gamma)$. Then by the above and the First Isomorphism Theorem, $G_\gamma/\ker\psi \cong \mathrm{Sym}(4)$. Since $\langle t_\gamma\rangle \le \ker\psi$, and $G_\gamma/Q_\gamma \cong \mathrm{GL}_2(3)$, this implies that $\ker\psi = Q_\gamma\langle t_\gamma\rangle$. $\square$

Lemma 5.1.5 *Θ has valency 2. In particular, Θ is a circuit.*

Proof. Since $\alpha \in \Theta$ by definition, we first consider $\Gamma(\alpha)$. Let $\overline{T} = TQ_\alpha\langle t_\alpha\rangle/Q_\alpha\langle t_\alpha\rangle \le G_\alpha/Q_\alpha\langle t_\alpha\rangle$. Then $\overline{T}$ has order 2 and so by Lemma 5.1.4, we have that $\overline{T}$ corresponds to an involution in $\mathrm{Sym}(4)$. Since $\overline{T} \not\le O_2(G_\alpha/Q_\alpha\langle t_\alpha\rangle)$, we have that $\overline{T}$ corresponds to a transposition. However, T fixes the edge $\{\alpha,\beta\}$, and hence T must fix another element of $\Gamma(\alpha)$. We can apply this argument repeatedly to show that Θ has valency 2. Since Θ is a finite subgraph and recalling that Θ is connected by definition, we see that we will eventually return to the vertex β, hence forming a circuit. $\square$

Lemma 5.1.6 *Suppose that $\gamma \in \Gamma_\beta$. Then $C_{G_\gamma}(C_\gamma/W_\gamma) \ge L_\gamma$, $C_{G_\gamma}(Z(W_\gamma)/V_\gamma) \ge L_\gamma$ and $C_{G_\gamma}(Z_\gamma) \ge L_\gamma$.*

Proof. By Definition 1.6.3, C_γ/W_γ, $Z(W_\gamma)/V_\gamma$ and Z_γ are trivial L_γ/Q_γ-modules and are centralized by L_γ. Hence the result follows. $\square$

Lemma 5.1.7 *Suppose $\gamma \in \Gamma_\beta$. Then $C_{Q_\gamma}(t_\gamma)$ is an extra-special group of order 3^3.*

Proof. Since t_γ is in L_γ, by Lemma 5.1.6, t_γ centralizes C_γ/W_γ, $Z(W_\gamma)/V_\gamma$ and Z_γ. We also have that t_γ inverts Q_γ/C_γ, $W_\gamma/Z(W_\gamma)$ and V_γ/Z_γ. Hence, we repeatedly apply Coprime Action (*iv*) to see that $|C_{Q_\gamma}(t_\gamma)| = 3^3$.

We have that $C_{Q_\gamma}(t_\gamma) \le C_\gamma$. However, $C_{Q_\gamma}(t_\gamma) \not\le W_\gamma$ since $[C_\gamma, t_\gamma] \le W_\gamma$. Therefore, $C_\gamma = C_{Q_\gamma} t_\gamma W_\gamma$ and $(C_{Q_\gamma}(t_\gamma) \cap Z(W_\gamma)V_\gamma = Z(W_\gamma)$.

Suppose $C_{Q_\gamma}(t_\gamma)$ is abelian. Then $C_{Q_\gamma}(t_\gamma)$ centralizes $(C_{Q_\gamma}(t_\gamma) \cap Z(W_\gamma)Z(C_\gamma) = (C_{Q_\gamma}(t_\gamma) \cap Z(W_\gamma)V_\gamma = Z(W_\gamma)$, by Lemma 4.2.1, (vii) and the above. Since W_γ centralizes $Z(W_\gamma)$ by definition, we have that $C_{Q_\gamma}(t_\gamma)W_\gamma = C_\gamma$ centralizes $Z(W_\gamma)$. Therefore, $Z(C_\gamma) = Z(W_\gamma)$, a contradiction. Hence $C_{Q_\gamma}(t_\gamma)$ is an extra-special group. $\square$

Lemma 5.1.8 *Suppose $\gamma \in \Gamma_\beta$. Then $O_2(C_{L_\gamma}(t_\gamma)) \in \mathrm{Syl}_2(C_{L_\gamma}(t_\gamma))$ and $O_2(C_{L_\gamma}(t_\gamma)) \cong$ Q_8.*

Proof. Suppose $M_\gamma = O_{3,2}(L_\gamma)$. Then

$$M_\gamma/Q_\gamma = O_2(L_\gamma/Q_\gamma) \cong O_2(\mathrm{SL}_2(3)) \cong \mathrm{Q}_8.$$

Let $N_\gamma \in \mathrm{Syl}_2(M_\gamma)$ such that $\langle t_\gamma \rangle \leq N_\gamma$. Then $N_\gamma Q_\gamma = M_\gamma$ and $N_\gamma \cap Q_\gamma = 1$. So,

$$N_\gamma \cong N_\gamma Q_\gamma/Q_\gamma = M_\gamma/Q_\gamma = \mathrm{Q}_8.$$

Also, $N_\gamma \leq C_{L_\gamma}(t_\gamma)$. Since $N_\gamma \in \mathrm{Syl}_2(M_\gamma)$, we have that $N_\gamma \in \mathrm{Syl}_2(C_{L_\gamma}(t_\gamma))$. We claim $N_\gamma \trianglelefteq C_{M_\gamma}(t_\gamma)$. Since $M_\gamma = N_\gamma Q_\gamma$, we have that $C_{M_\gamma}(t_\gamma) = C_{Q_\gamma}(t_\gamma)C_{N_\gamma}(t_\gamma) = C_{Q_\gamma}(t_\gamma)N_\gamma$, as $t_\gamma \in N_\gamma$. By Lemma 5.1.7, $|C_{Q_\gamma}(t_\gamma)| = 3^3$. Since $L_\gamma \geq N_\gamma$, by Lemma 5.1.6, N_γ centralizes C_γ/W_γ, $Z(W_\gamma)/V_\gamma$ and Z_γ. Also, t_γ inverts Q_γ/C_γ, $W_\gamma/Z(W_\gamma)$ and V_γ/Z_γ. Therefore, we apply Coprime Action (iv) repeatedly to see that $|C_{Q_\gamma}(N_\gamma)| = 3^3$. Therefore $C_{Q_\gamma}(t_\gamma) = C_{Q_\gamma}(N_\gamma)$. Hence N_γ is centralized by $C_{Q_\gamma}(t_\gamma)$ and therefore $N_\gamma \trianglelefteq C_{Q_\gamma}(t_\gamma)N_\gamma = C_{M_\gamma}(t_\gamma)$. Since $N_\gamma \in \mathrm{Syl}_2(C_{M_\gamma}(t_\gamma))$, this implies that N_γ is the unique subgroup of $C_{M_\gamma}(t_\gamma)$ of order 8 and hence N_γ is a characteristic subgroup of $C_{M_\gamma}(t_\gamma)$. Therefore, since $|L_\gamma : M_\gamma| = 3$ and $M_\gamma \trianglelefteq L_\gamma$, we see that $C_{M_\gamma}(t_\gamma) \trianglelefteq C_{L_\gamma}(t_\gamma)$. Hence $N_\gamma \trianglelefteq C_{L_\gamma}(t_\gamma)$. Therefore $O_2(C_{L_\gamma}(t_\gamma)) = N_\gamma \cong \mathrm{Q}_8$ and $O_2(C_{L_\gamma}(t_\gamma)) \in \mathrm{Syl}_2(C_{L_\gamma}(t_\gamma))$, completing the proof of the result. $\square$

Definition 5.1.9 Let $\gamma \in \Theta_\beta$ and $(\gamma-2, \gamma-1, \gamma, \gamma+1, \gamma+2)$ be a path of length 4 in Θ. Define

$$P_\gamma = \langle W_{\gamma-2}, W_\gamma, W_{\gamma+2} \rangle T.$$

We also define

$$Y = \bigcap_{\theta \in \Theta_\beta} Z(W_\theta),$$

which we show is the subgroup in the conclusions of Theorem A.

Lemma 5.1.10 *Suppose that $(\gamma, \gamma+1)$ is a path in Θ, with $\gamma \in \Theta_\beta$. Then $(\gamma, \gamma+1, \gamma+2)$ can be extended uniquely to a path $(\ldots, \gamma-6, \gamma-5, \ldots, \gamma, \gamma+1, \gamma+2, \ldots, \gamma+6, \ldots)$ of any finite length.*

Proof. This follows since Θ is a circuit by Lemma 5.1.5. $\square$

The following results are Proposition 13.4 from [32].

Lemma 5.1.11 *Suppose that $\gamma \in \Theta_\beta$ and let $X_\gamma = \langle Z_{\gamma-6}, Z_{\gamma+6} \rangle$. Then:*

(i) $P_\gamma = TX_\gamma W_\gamma$;

(ii) $P_\gamma / W_\gamma \cong TX_\gamma \cong \mathrm{GL}_2(3)$;

(iii) $P_\gamma \cap P_{\gamma+2} = TW_\gamma W_{\gamma+2}$; *and*

(iv) $|P_\gamma| = |P_{\gamma+2}| = 2^4 3^7$.

Proof. Since $\gamma-6, \gamma+6 \in \Theta_\beta$ we have that $Z_{\gamma-6}$ and $Z_{\gamma+6}$ both have order 3. By Lemma 5.1.1 (*iv*), $t_{\gamma-6} = t_\gamma = t_{\gamma+6}$. So X_γ centralizes t_γ, since $Z_{\gamma-6}$ and $Z_{\gamma+6}$ centralize $t_{\gamma-6} = t_\gamma$ and $t_{\gamma+6} = t_\gamma$ respectively. By Lemma 5.1.1 (*i*) and (*ii*), we have that the critical distance of Γ, $b = 5$ and that G acts locally 7-arc transitively on Γ. Hence,

$$Z_{\gamma-6} \leq Z_{\gamma-5} \leq Q_{\gamma-1} \leq L_\gamma.$$

Similarly,

$$Z_{\gamma+6} \leq Z_{\gamma+5} \leq Q_{\gamma+1} \leq L_\gamma.$$

Therefore $X_\gamma \leq L_\gamma$ and so $X_\gamma \leq C_{L_\gamma}(t_\gamma)$. By Lemma 5.1.8, $O_2(C_{L_\gamma}(t_\gamma)) \cong \mathrm{Q}_8$ since $\gamma \in \Gamma_\beta$. We consider the structure of L_γ, as given in Definition 1.6.3. Since X_γ acts transitively on $\Gamma(\gamma)$, we see that $X_\gamma Q_\gamma/Q_\gamma \cong \mathrm{SL}_2(3)$ and therefore $X_\gamma \geq O_2(C_{L_\gamma}(t_\gamma)) \cong \mathrm{Q}_8$. Clearly $Z_{\gamma-6}O_2(C_{L_\gamma}(t_\gamma)) \cong \mathrm{SL}_2(3)$. We show $Z_{\gamma+6} \leq Z_{\gamma-6}O_2(C_{L_\gamma}(t_\gamma))$. Let $E = O_2(C_{L_\gamma}(t_\gamma))$. Then $E_0 = N_E(T) = \langle t_\gamma, x\rangle$ where x has order 4 and $x^2 = t_{\gamma+1}$. So x stabilizes Θ, and therefore $Z_{\gamma-6}^x = Z_{\gamma+6}$. Therefore $Z_{\gamma-6}O_2(C_{L_\gamma}(t_\gamma)) \geq Z_{\gamma-6}^x = Z_{\gamma+6}$. Hence we have that $X_\gamma = O_2(C_{L_\gamma}(t_\gamma))Z_{\gamma-6} \cong \mathrm{SL}_2(3)$. Since T normalizes X_γ and inverts $Z_{\gamma-6}$, we have that $TX_\gamma \cong \mathrm{GL}_2(3)$. Therefore, it remains to show that $P_\gamma = TX_\gamma W_\gamma$ as X_γ normalizes W_γ. Since $W_{\gamma-2}$ and W_γ both contain $U_{\gamma+1}$ and $W_{\gamma-2} \neq W_\gamma$, we have that $W_{\gamma-2} \cap W_\gamma = U_{\gamma+1}$. Similarly, $W_{\gamma-2} \cap W_{\gamma-4} = U_{\gamma-5} \geq Z_{\gamma-5} \geq Z_{\gamma-6}$. Hence $Z_{\gamma-6} \leq W_{\gamma-2}$. We also have that $Z_{\gamma-6} \not\leq Q_\gamma$ since b is 5 and G acts locally 7-arc transitively on Γ by Lemma 5.1.1 (i) and (ii). So, $W_{\gamma-2} = Z_{\gamma-6}U_{\gamma+1} = Z_{\gamma-6}(W_{\gamma-2} \cap W_\gamma)$ and similarly, $W_{\gamma+2} = Z_{\gamma+6}U_{\gamma-1} = Z_{\gamma+6}(W_{\gamma+2} \cap W_\gamma)$. So,

$$P_\gamma = \langle Z_{\gamma-6}, Z_{\gamma+6}\rangle W_\gamma T = X_\gamma W_\gamma T,$$

and by shifting to vertex $\gamma + 2$,

$$P_{\gamma+2} = X_{\gamma+2}W_{\gamma+2}T,$$

where $X_{\gamma+2} = \langle Z_{\gamma-4}, Z_{\gamma+8}\rangle$. Therefore (i) holds. Since X_γ normalizes W_γ, (ii) also holds. As $|\mathrm{GL}_2(3)| = 2^4 3$ and $|W_\gamma| = 3^6$, we see that (iv) follows from (ii).

We have that $P_\gamma \cap P_{\gamma+2} = TX_\gamma W_\gamma \cap TX_{\gamma+2}W_{\gamma+2} \supseteq W_\gamma W_{\gamma+2}T$. Since $P_\gamma \neq P_{\gamma+2}$ and $W_\gamma W_{\gamma+2}T$ is maximal of index four in both P_γ and $P_{\gamma+2}$, we see that equality holds in the above and hence (iii) follows. $\square$

Since for $\gamma \in \Theta_\beta$, P_γ is G-conjugate to P_β, we may assume without loss of generality that $\gamma = \beta$. Therefore, for the rest of this chapter we consider the path $\Pi = (\ldots \beta-2, \beta-1 = \alpha, \beta, \beta+1, \beta+2, \ldots)$. This is the unique extension of the path $(\beta-1 = \alpha, \beta, \beta+1)$ in Θ by Lemma 5.1.10.

Lemma 5.1.12 *$T \leq C_G(Y)$. In particular, Y is centralized by P_β and $P_{\beta+2}$.*

Proof. Let $X_\gamma = \langle Z_{\gamma-6}, Z_{\gamma+6}\rangle$ for $\gamma \in \Theta_\beta$, as in Lemma 5.1.11. We have that $[X_\beta, Y] = 1$ and $[X_{\beta+2}, Y] = 1$ since $Y \leq Z(W_{\beta-6}) \cap Z(W_{\beta+6})$ and $Y \leq Z(W_{\beta-4}) \cap Z(W_{\beta+8})$. Also X_β centralizes t_β and $X_{\beta+2}$ centralizes $t_{\beta+2}$. Therefore, since $t_\beta \leq X_\beta$ and $t_{\beta+2} \leq X_{\beta+2}$, we have that $[t_\beta, Y] = [t_{\beta+2}, Y] = 1$. So $t_\beta, t_{\beta+2} \in C_G(Y)$. Since $T = \langle t_\beta, t_{\beta+2}\rangle$, this implies that T centralizes Y and hence P_β centralizes Y.

We have that $P_\beta = \langle W_{\beta-2}, W_\beta, W_{\beta+2}\rangle T$. Since $Z(W_\delta) \leq Y$ for $\delta \in \{\beta-2, \beta, \beta+2\}$, $W_\delta \leq C_G(Y)$. Therefore P_β centralizes Y. Similarly, $P_{\beta+2}$ centralizes Y. $\square$

We note that if $\gamma \in \Gamma(\alpha)$, then since G_γ is conjugate to G_β, we have that $Z_\gamma \leq Z_\alpha$.

Lemma 5.1.13 *$|Y| \leq 3$.*

Proof. Since $Z(W_\beta) \neq Z(W_{\beta+2})$ and $U_\alpha = Z(W_\beta)Z(W_{\beta+2})$, we have that $|Z(W_\beta) \cap Z(W_{\beta+2})| = 3^3$. By definition $Y \leq Z(W_\beta) \cap Z(W_{\beta+2})$ and we also have that $Z_\alpha \leq Z(W_\beta) \cap Z(W_{\beta+2})$. Suppose that $|Y| \geq 3^2$. Then since $|Z_\alpha| = 3^2$ (see Definition 1.6.3), we have that $Y \cap Z_\alpha \neq 1$. So, there exists $\theta \in \Gamma(\alpha)$ such that $Y \cap Z_\alpha \geq Z_\theta$ and therefore $Z_\theta \leq Y$. Choose $\delta \in \{\beta, \beta+2\}$ such that $\delta \neq \theta$. Then, by Lemma 5.1.12 and our global hypothesis that $N_G(Z_\beta) = G_\beta$, $P_\delta \leq C_G(Y) \leq C_G(Z_\theta) = G_\theta$. Since $P_\delta \leq G_\delta$, this implies that $P_\delta \in G_\delta \cap G_\theta$. We have that, $|P_\delta|_2 = 2^4$ by Lemma 5.1.11 (iv). Since by Lemma 5.1.1, $b = 5$ and G acts locally 7-arc transitively on Γ, we have that Γ does not contain any 4-cycles. Hence, $G_\delta \cap G_\theta \leq G_{\alpha\beta}$. So $|G_\delta \cap G_\theta|_2 \leq |G_{\alpha\beta}|_2 = 2^2$. Therefore we have a contradiction and hence $|Y| \leq 3$. $\square$

Lemma 5.1.14 *Y is the largest subgroup of $W_\beta W_{\beta+2}$ that is also normalized by $N_G(Y)$.*

Proof. Suppose that K is the largest normal subgroup of $N_G(Y)$ that is also contained in $W_\beta W_{\beta+2}$. Then by Lemma 5.1.11 (ii), $K \leq U_\alpha = W_\beta \cap W_{\beta+2} = Z(W_\beta)Z(W_{\beta+2})$. Since P_β acts transitively on the neighbours of β, P_β cannot normalize U_α. Therefore, since $K \trianglelefteq P_\beta$, this implies that $K \leq Z(W_\beta)$. We can repeat this argument for any vertex in Θ_β, and hence $K \leq \bigcap_{\theta \in \Theta_\beta} Z(W_\theta) = Y$. Since Y is normal in $W_\beta W_{\beta+2}$ the maximality of K implies that $K = Y$ and the result follows. $\square$

Lemma 5.1.15 *The amalgam $\mathcal{G} = \mathcal{G}(P_\beta/Y, P_{\beta+2}/Y, W_\beta W_{\beta+2}T/Y)$ is an amalgam of type* $\mathrm{G}_2(3)$*. In particular $|Y| = 3$.*

Proof. By Lemma 5.1.12, Y is centralized by P_δ for $\delta \in \{\beta, \beta+2\}$. Hence Y is centralized by $O^{3'}(P_\delta)$ and therefore Lemma 5.1.14 implies that $Y \cap V_\delta \leq Z_\delta$. So $O_3(P_\delta/Y)$ has two non-central $P_\delta/O_3(P_\delta)$-chief factors. We show that $\mathcal{G}$ is a weak BN-pair. To do this we show that $\mathcal{G}$ satisfies Definition 1.6.1. So $A_\delta = P_\delta/Y$ for $\delta \in \{\beta, \beta+2\}$ and $B = W_\beta W_{\beta+2}T/Y$. Let $A_\delta^* = O^{3'}(A_\delta)$. We have that $O_3(A_\delta) = O_3(P_\delta/Y) = W_\delta/Y$ and hence $O_3(A_\delta) \leq O^{3'}(P_\delta/Y) = A_\delta^*$. Since $W_\beta W_{\beta+2} \leq O^{3'}(P_\delta)$, we have that

$$A_\delta^* B = O^{3'}(P_\delta/Y)W_\beta W_{\beta+2}T/Y = O^{3'}(P_\delta/Y)T = P_\delta/Y = A_\delta.$$

Hence both parts of Definition 1.6.1 (i) are satisfied.

We have that P_δ/W_δ does not centralize W_δ/Y, and so

$$C_{A_\delta}(O_3(A_\delta)) = C_{P_\delta/Y}(O_3(P_\delta/Y)) = C_{P_\delta/Y}(W_\delta/Y) \leq W_\delta/Y = O_3(A_\delta),$$

and therefore Definition 1.6.1 (ii) holds.

By Lemma 5.1.11 (ii), $P_\delta/W_\delta \cong \mathrm{GL}_2(3)$. Hence

$$A_\delta^*/O_3(A_\delta) = O^{3'}(P_\delta/Y)/O_3(P_\delta/Y) \cong O^{3'}(P_\delta)/W_\delta \cong \mathrm{SL}_2(3).$$

By the proof of Lemma 5.1.11, $t_\delta \in \langle Z_{\delta-6}, Z_{\delta+6}\rangle \cong \mathrm{SL}_2(3)$. Since $X_\delta \leq O^{3'}(X_\delta) \leq O^{3'}(P_\delta)$, we therefore have that $t_\delta \in O^{3'}(P_\delta)$. Since t_δ has order 3, this implies that $t_\delta \in O^{3'}(P_\delta/Y) = A_\delta^*$. Let $\{\delta, \epsilon\} = \{\beta, \beta+2\}$. So,

$$A_\delta^* \cap B = O^{3'}(P_\delta/Y) \cap W_\beta W_{\beta+2}T/Y = \langle t_\delta\rangle W_\beta W_{\beta+2}/Y \leq T\langle W_\beta, W_{\beta+2}\rangle/Y \leq P_\epsilon/Y = A_\epsilon.$$

So, since $W_\beta W_{\beta+2}/Y$ is normal in P_ϵ/Y and $W_\beta W_{\beta+2}/Y \in \mathrm{Syl}_3(P_\epsilon/Y)$, we see that $A_\delta^* \cap B = O^{3'}(P_\delta/Y) \cap W_\beta W_{\beta+2}T/Y$ normalizes $W_\beta W_{\beta+2}/Y$ and hence Definition 1.6.1 (iii) is satisfied.

Therefore $\mathcal{G}$ is a weak BN-pair. By Lemma 5.1.13, $|Y| \leq 3$ and therefore $|P_\beta/Y|_3 = |P_{\beta+2}/Y|_3 \geq 3^6$. Since the main theorem in [9, Theorem A, page 100] gives possible orders for P_β/Y and $P_{\beta+2}/Y$, we see that $|P_\beta/Y|_3 = |P_{\beta+2}/Y|_3 = 3^6$. In particular $|P_\beta/Y|_3 = 3^6 = \frac{3^7}{|Y|}$ and therefore $|Y| = 3$.Hence we have an amalgam satisfying all the conditions in Definition 1.6.1 apart from (i). Hence, in order to show that $\mathcal{G}$ is an amalgam of type $\mathrm{G}_2(3)$, it remains to show that,

$$P_\delta/Y = (O^{3'}(P_\epsilon/Y) \cap W_\beta W_{\beta+2}T/Y)O^{3'}(P_\delta/Y),$$

for $\{\delta, \epsilon\} = \{\beta, \beta+2\}$. Since $O^{3'}(P_\epsilon) \cap W_\beta W_{\beta+2}T \leq P_\delta$ and $O^{3'}(P_\delta) \leq P_\delta$, we have that

$$(O^{3'}(P_\epsilon/Y) \cap W_\beta W_{\beta+2}T/Y)O^{3'}(P_\delta/Y) \leq P_\epsilon/Y.$$

However, $t_\theta \in O^{3'}(P_\theta/Y)$ for $\theta \in \{\delta, \epsilon\}$, and hence, as $T = \langle t_\delta, t_\epsilon\rangle$, we have that $P_\delta/Y =$

$(O^{3'}(P_\epsilon/Y) \cap W_\beta W_{\beta+2}T/Y)O^{3'}(P_\delta/Y)$, as required.

Hence $\mathcal{G}$ is an amalgam of type $\mathrm{G}_2(3)$. $\square$

Let $Q = O_3(N_G(Y))$. We aim to show that $Q = Y$. The following result is used many times in the rest of this chapter and Chapter 6.

Lemma 5.1.16 *(i)* $|N_{N_G(Y)}(W_\beta W_{\beta+2}) : N_{C_G(Y)}(W_\beta W_{\beta+2})| = 2$. *In particular,* $|N_G(Y) : C_G(Y)| = 2$ *and there exists* $x \in G$ *that inverts* Y *and permutes the set* $\{W_\beta, W_{\beta+2}\}$.

(ii) $N_{C_G(Y)}(W_\beta W_{\beta+2}) \leq P_\beta \cap P_{\beta+2}$.

(iii) There exists a conjugate of t_β *in* G *which inverts* Y.

Proof. (i) By Lemma 4.1.3 (i) and (iii), W_β and $W_{\beta+2}$ are the unique subgroups of $W_\beta W_{\beta+2}$ that have exponent 3. Hence, if $x \in N_{N_G(Y)}(W_\beta W_{\beta+2})$, either $x \in N_{N_G(Y)}(W_\beta) \cap N_{N_G(Y)}(W_{\beta+2}) \leq N_G(W_\beta) \cap N_G(W_{\beta+2})$ and hence either x does not interchange W_β and $W_{\beta+2}$, or x permutes the set $\{W_\beta, W_{\beta+2}\}$. First suppose that $x \in N_{N_G(Y)}(W_\beta W_{\beta+2})$ does not interchange W_β and $W_{\beta+2}$. Let $\delta \in \{\beta, \beta+2\}$. Then $N_G(W_\delta) \leq N_G(Z_\delta) \leq G_\delta$ since $W_\delta' = Z_\delta$ by Lemma 4.2.2 (ii). Hence $x \in N_{G_\delta}(Y) = P_\delta$. Then by Lemma 5.1.12, we have that $x \in C_G(Y)$ and so $x \in N_{C_G(Y)}(W_\beta W_{\beta+2})$. Hence $|N_{N_G(Y)}(W_\beta W_{\beta+2}) : N_{C_G(Y)}(W_\beta W_{\beta+2})| \leq 2$.

Suppose $R \in \mathrm{Syl}_2(L_\alpha)$ such that $T \leq R$. Then R is isomorphic to a Sylow 2-subgroup of $GL_\alpha/Q_\alpha \cong \mathrm{SL}_2(3)$ and therefore, by [6, page 142, I], $R \cong \mathrm{SDih}(8)$. We have that $T = \langle t_\alpha, t_\beta \rangle$, where t_α and t_β are involutions by Lemma 5.1.1. Let $T^* = N_R(T)$. Then $T^* \cong \mathrm{Dih}(8)$. Suppose that $t^* \in T^*$ such that t^* has order 4, $(t^*)^2 = t_\alpha$ and $(t^*)^{t_\beta} = (t^*)^{-1}$. Then $T^* \cong \langle t^*, t_\beta \rangle$. We can consider T as the subgroup of $\mathrm{GL}_2(3)$ generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, we see that $\det(t^*) = 1$. Since t^* acts on U_α/Z_α, this implies that t^* inverts a 2-space and centralizes a 1-space. We see that t_β centralizes $Z(W_\beta)/V_\beta$ and inverts $U_\alpha/Z(W_\beta)$ and V_β/Z_α. Hence we can decompose U_α/Z_α as,

$$U_\alpha/Z_\alpha = [U_\alpha/Z_\alpha, t_\beta] \oplus C_{U_\alpha/Z_\alpha}(t_\beta) = [U_\alpha/Z_\alpha, t_\beta] \oplus YZ_\alpha/Z_\alpha.$$

Since t^* and t_β commute modulo t_α, we see that t^* preserves this decomposition. We claim t^* inverts Y. Suppose that t^* centralizes YZ_α/Z_α. Then $C_{U_\alpha/Z_\alpha}(t^*) = C_{U_\alpha/Z_\alpha}(t_\beta)$ and therefore t^* and t_β invert $[U_\alpha/Z_\alpha, t_\beta]$. So t^*t_β centralizes $[U_\alpha/Z_\alpha, t_\beta]$ which implies that t^*t_β centralizes U_α/Z_α. This is a contradiction and hence t^* inverts YZ_α/Z_α. Since Y is centralized by t_α and $(t^*)^2 = t_\alpha$, we see that t^* normalizes Y. Hence Y is inverted by t^*. This implies that $|N_G(Y) : C_G(Y)| = 2$. Therefore, the elements of $N_{N_G(Y)}(W_\beta W_{\beta+2})$ which do not interchange W_β and W_β are in $C_G(Y)$, and there exists $x \in N_{N_G(Y)}(W_\beta W_{\beta+2})$ which interchanges W_β and $W_{\beta+2}$. Since $N_G(Y) \neq C_G(Y)$ and t^* inverts Y, we can write

$$N_G(Y) = C_G(Y) \dot{\cup} C_G(Y)t^*,$$

and we see that $x \in C_G(Y)t^*$. So $x = ct^*$ for some $c \in C_G(Y)$ and therefore, for $y \in Y$,

$$y^x = y^{ct^*} = y^{t^*} = y^{-1},$$

and hence x inverts Y. Therefore, $x \in G$ inverts Y and permutes the set $\{W_\beta, W_{\beta+2}\}$.

(*ii*) Let $F = N_{N_G(Y)}(W_\beta W_{\beta+2})$. Then since $C_F(Y) = N_{C_G(Y)}(W_\beta W_{\beta+2})$, by (*i*), we have that $|F : C_F(Y)| = 2$. Clearly $C_F(Y) \trianglelefteq F$ since $C_G(Y) \trianglelefteq N_G(Y)$. Also by (*i*), there exists $x \in F$ that interchanges W_β and $W_{\beta+2}$ and therefore we see that F acts transitively on $\{W_\beta, W_{\beta+2}\}$. Let $\phi : F \to \mathrm{Sym}(\{W_\beta, W_{\beta+2}\})$. Then ϕ is surjective. Consider $N_F(W_\beta)$. This fixes W_β, and hence also fixes $W_{\beta+2}$ and

$|F : N_F(W_\beta)| = 2$. Therefore $N_F(W_\beta) \cong \ker \phi$ and hence is a normal subgroup of F. So, we have two normal subgroups of F of index 2, namely $C_F(Y)$ and $N_F(W_\beta)$. Since $N_F(W_\beta) \leq N_G(W_\beta)$, we see from the proof of (i) that $N_F(W_\beta) \leq C_F(Y)$ and hence, by orders, $N_F(W_\beta) = C_F(Y)$. Similarly, since $N_F(W_{\beta+2})$ is also a normal subgroup of F of index 2, $N_F(W_{\beta+2}) = C_F(Y)$. We have that $N_F(W_\beta) \leq P_\beta$ and $N_F(W_{\beta+2}) \leq P_{\beta+2}$. Therefore,

$$N_{C_G(Y)}(W_\beta W_{\beta+2}) = C_F(Y) \leq P_\beta \cap P_{\beta+2}.$$

(iii) Let R, T^* and t^* be as in the proof of (i). Then T^* contains three subgroups of order 4 which contain $Z(T^*) = \langle t_\alpha \rangle$, namely two distinct fours groups T and T_1 and a cyclic group C of order 4. Clearly $C = \langle t^* \rangle$ since t^* has order 4. Since $(t^*)^{t_\beta} = (t^*)^{-1}$, we see that

$$(t_\beta t^*)^2 = t_\beta t^* t_\beta t^* = (t^*)^{-1} t^*,$$

and hence $t_\beta t^*$ is an involution. As t_β centralizes Y and t^* inverts Y, we have that $t_\beta t^*$ inverts Y and hence $t_\beta t^* \notin T$ since $T \leq C_G(Y)$ by Lemma 5.1.12. Therefore $t_\beta t^* \in T_1$ and so $T_1 = \{1, t_\alpha, t_\beta t^*, t_\alpha t_\beta t^* = t_{\beta+2} t^*\} = \langle t_\alpha, t_\beta t^* \rangle$. It remains to show that $t_\beta t^*$ is G-conjugate to t_β. Now $t_\beta t^* \in G_\alpha \cap G_\gamma$ where $\gamma \in \Gamma(\alpha) \backslash \Theta$ and $t_\beta t^*$ commutes with t_α since T_1 is a fours group. Therefore T_1 is a complement to $S_{\alpha\gamma}$ in $G_{\alpha\gamma}$. Hence, the elements of T_1 are G-conjugate by Lemma 5.1.1 (vi). In particular $t_\beta t^*$ is G-conjugate to t_α. As t_α is G-conjugate to t_β, again by Lemma 5.1.1 (vi), this implies that $t_\beta t^*$ is G-conjugate to t_β, completing the proof of the result. □

Lemma 5.1.17 *$W_\beta W_{\beta+2} \in \mathrm{Syl}_3(N_G(Y))$.*

Proof. Suppose that $W_\beta W_{\beta+2} \notin \mathrm{Syl}_3(N_G(Y))$ and let $K \in \mathrm{Syl}_3(N_G(Y))$ such that $K > W_\beta W_{\beta+2}$. By Lemma 5.1.16 (i), $C_G(Y)$ has index 2 in $N_G(Y)$, and hence $K \in \mathrm{Syl}_3(C_G(Y))$, Let $K_0 = N_K(W_\beta W_{\beta+2})$. Hence $W_\beta W_{\beta+2} < K_0$ and $|K_0| = 3^n$ where $n > 7$. By Lemma 5.1.16 (ii), $K_0 \leq N_{C_G(Y)}(W_\beta W_\beta) \leq P_\beta \cap P_{\beta+2}$. So $|K_0| \leq |P_\beta \cap P_{\beta+2}|_3 = 3^7$, a contradiction. Hence $W_\beta W_{\beta+2} \in \mathrm{Syl}_3(N_G(Y))$. □

Lemma 5.1.18 *Q is equal to Y.*

Proof. Since $W_\beta W_\beta \in \mathrm{Syl}_3(N_G(Y))$ by Lemma 5.1.17 this implies that $Q \leq W_\beta W_{\beta+2}$. Therefore, by Lemma 5.1.14 this implies that $Q = Y$. □

5.2 The Simplicity of $C_G(Y)/Y$

We show that $C_G(Y)/Y$ is a non-abelian simple group. This will enable us to show that $C_G(Y)/Y \cong \mathrm{G}_2(3)$, using our $\mathcal{K}$-proper hypothesis and the results in Chapter 2.

We first define some notation. For a subgroup H of G such that $Y \trianglelefteq H$, let

$$\overline{H} = H/Y.$$

Lemma 5.2.1 $O_{3'}(\overline{C_G(Y)}) = \langle C_{O_{3'}(\overline{C_G(Y)})}(x) \mid x \in Z_\alpha^\# \rangle$.

Proof. Since Z_α is elementary abelian of order 9, this follows by Coprime Action. □

Lemma 5.2.2 $O_{3'}(P_\delta) = 1$ *for* $\delta \in \{\beta, \beta+2\}$.

Proof. Let $\delta \in \{\beta, \beta+2\}$. By definition, $O_{3'}(P_\delta) \trianglelefteq P_\delta$ and $O_3(P_\delta) \trianglelefteq P_\delta$. Hence,

$$[O_3(P_\delta), O_{3'}(P_\delta)] \leq O_3(P_\delta) \cap O_{3'}(P_\delta) = 1.$$

Since $C_{P_\delta/Y}(O_3(P_\delta/Y)) \leq O_3(P_\delta/Y)$, as P_δ/Y is of characteristic 3, this implies that $O_{3'}(P_\delta) = 1$. □

Lemma 5.2.3 *Let $\theta \in \Gamma(\alpha)$, $O_{3'}(N_{G_\theta}(Y)) = 1$.*

Proof. We have that $|\Gamma(\alpha)| = 4$ by Lemma 5.1.1 (vii) and $\beta, \beta+2 \in \Gamma(\alpha)$ and $N_{G_\delta}(Y) = P_\delta$ for $\delta \in \{\beta, \beta+2\}$. So, since $O_{3'}(P_\beta) = O_{3'}(P_{\beta+2}) = 1$ by Lemma 5.2.2, we are done in these cases. So suppose that $\theta \in \Gamma(\alpha)\backslash\Theta$. By Lemma 5.1.17, $W_\beta W_{\beta+2} \in \mathrm{Syl}_3(N_G(Y))$. Since $W_\beta W_{\beta+2} \leq Q_\alpha \leq G_\theta$, this implies that $W_\beta W_{\beta+2} \in \mathrm{Syl}_3(N_{G_\theta}(Y))$. We claim that $W_\beta W_{\beta+2}$ is the unique Sylow 3-subgroup of $N_{G_\theta}(Y)$. Suppose that there exists $F \in \mathrm{Syl}_3(N_{G_\theta}(Y))$ such that $F \neq W_\beta W_{\beta+2}$. Then $\langle F, W_\beta W_{\beta+2}\rangle Q_\theta = L_\theta$. Therefore,

$$\langle F, W_\beta W_{\beta+2}\rangle Q_\beta/Q_\beta \cong L_\theta/Q_\theta \cong \mathrm{SL}_2(3).$$

By Lemma 5.1.16 (iii), there exists an involution $t \in t_\theta Q_\theta$ which inverts Y. So there exists $t^{**} \in \langle F, W_\beta W_{\beta+2}\rangle$ such that $t^{**}Q_\theta = tQ_\theta$ and t^{**} centralizes Y. So $t^{**}t$ inverts Y. However, $t^{**}t \in Q_\theta$, and hence does not invert Y and therefore this is a contradiction. Hence $W_\beta W_{\beta+2} \trianglelefteq N_{G_\theta}(Y)$ and thus $O_{3'}(N_{G_\theta}(Y))$ centralizes $W_\beta W_{\beta+2}$. However, since $W_\beta W_{\beta+2}$ is a 3-group which centralizes V_θ, we see that $C_{G_\theta}(W_\beta W_{\beta+2}) \leq C_{G_\theta}(V_\theta) = C_\theta$. Hence $C_{G_\theta}(W_\beta W_{\beta+2})$ is a 3-group and therefore $O_{3'}(N_{G_\theta}(Y)) = 1$. □

Lemma 5.2.4 $O_{3'}(\overline{C_G(Y)}) = 1$

Proof. Suppose that $\gamma \in \Theta\backslash\Theta_\beta$ and $\theta \in \Gamma(\gamma)$. Let $x \in Z_\gamma^\#$ and consider $X := C_{O_{3'}(C_G(Y))}(x)$. Clearly $X \leq O_{3'}(N_G(Y))$. Since $Z_\theta \leq Y$, we have that $X \leq O_{3'}(C_G(Z_\theta)) \leq O_{3'}(N_G(Z_\theta)) = O_{3'}(G_\theta)$ and therefore,

$$X \leq O_{3'}(N_G(Y)) \cap O_{3'}(N_{G_\theta}(Y)).$$

So $\overline{X} = C_{O_{3'}(\overline{C_G(Y)})}(x) \leq O_{3'}(\overline{N_{G_\theta}(Y)})$. By Lemma 5.2.3, $O_{3'}(\overline{N_{G_\theta}(Y)}) = 1$ and hence $C_{O_{3'}(\overline{C_G(Y)})}(x) = 1$. Since x was arbitrarily chosen, this holds for all $x \in Z_\gamma^\#$ and thus $O_{3'}(\overline{C_G(Y)}) = 1$ by Lemma 5.2.1. □

Lemma 5.2.5 *$\overline{C_G(Y)}$ is a non-abelian simple group.*

Proof. We have that $\mathcal{G}$ is a 3-generated amalgam by Lemma 1.7.2 since it is an amalgam of type $\mathrm{G}_2(3)$ by Lemma 5.1.15. By Lemmas 5.1.12, 5.1.16 (*ii*) and 5.2.4 we have that $\langle\overline{P_\beta}, \overline{P_{\beta+2}}\rangle \le \overline{C_G(Y)}$, $N_{\overline{C_G(Y)}}(\overline{W_\beta W_{\beta+2}}) \le \overline{P_\beta} \cap \overline{P_{\beta+2}} \le \langle\overline{P_\beta}, \overline{P_{\beta+2}}\rangle$ and $O_{3'}(\overline{C_G(Y)}) = 1$. Also, by Lemma 5.1.17, $\mathrm{Syl}_3(\overline{W_\beta W_{\beta+2} T}) \subseteq \mathrm{Syl}_3(\overline{C_G(Y)})$. Therefore, the result follows from Theorem 1.7.3 with $\mathcal{A} = \mathcal{G}$ and $H = \overline{C_G(Y)}$. $\square$

Lemma 5.2.6 $\overline{C_G(Y)} \cong \mathrm{G}_2(3)$.

Proof. We see from Definition 1.6.2 and Lemmas 4.1.1, 4.1.2 and 4.1.3 and 5.1.17 that $\overline{C_G(Y)}$ satisfies the conditions in Hypothesis 2.0.1. Hence $\overline{C_G(Y)} \cong \mathrm{G}_2(3)$ by Theorem 2.0.2. $\square$

5.3 Completing the Proof of Theorem A

We now proceed to show that $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$. We have that $\overline{C_G(Y)} \cong \mathrm{G}_2(3)$ by Lemma 5.2.6 and so $C_G(Y) \cong Y.\mathrm{G}_2(3)$. Hence $|C_G(Y)| = |Y||\mathrm{G}_2(3)|$ and also $|N_G(Y) : C_G(Y)| = 2$, by Lemma 5.1.16. Therefore, the possibilities for $N_G(Y)$ are:

A. $\mathrm{Sym}(3) \times \mathrm{G}_2(3)$;

B. $6 \times \mathrm{G}_2(3)$;

C. $3 \times \mathrm{Aut}(\mathrm{G}_2(3))$;

D. $3^{\cdot}\mathrm{G}_2(3) : 2$; and

E. $(3 \times \mathrm{G}_2(3)) : 2$.

By Lemma 5.1.16 (*i*), there exists $x \in N_G(Y)$ that inverts Y and interchanges W_β and $W_{\beta+2}$. Since for $\delta \in \{\beta, \delta\}$, $W_\delta = O_3(P_\delta)$ where P_δ is a maximal parabolic subgroup of $N_G(Y)$, we see that cases A, B and C cannot occur.

In order to prove Theorem 5.3.3 below we require two further results, one about our amalgam of type $\mathrm{G}_2(3)$ and the other a known fact about $3^{\cdot}\mathrm{G}_2(3)$.

Lemma 5.3.1 *Let Y be the subgroup defined in Section 5.1 and $\mathcal{G}$ be the amalgam of type* $\mathrm{G}_2(3)$ *defined in Lemma 5.1.15. Then $Y \nleq W'_\beta$ and $Y \nleq W'_{\beta+2}$.*

Proof. We prove the result for W_β. The other case is identical due to the evident symmetric nature of an amalgam of type $\mathrm{G}_2(3)$, see Definition 1.6.2.

Since by Lemma 4.2.2 (*ii*), $W'_\beta = Z_\beta$, we show that $Y \nleq Z_\beta$. We have that $|Y| = |Z_\beta| = 3$ since $\beta \in \Gamma_\beta$. Therefore, if $Y \leq Z_\beta$, then $Y = Z_\beta$. This is a contradiction since Z_β is 3-central in G, and clearly Y is not. Hence $Y \nleq W'_\beta$ as required. □

Lemma 5.3.2 *Suppose that $K \cong 3^{\cdot}\mathrm{G}_2(3)$. Then $Z(K) \leq (O_3(P))'$, where P is a maximal parabolic subgroup of K.*

Proof. We prove this result by considering the permutation representation of K on 1134 points. We obtain generators for the maximal parabolic subgroups given by the online Atlas of Finite Groups, [5] and then use a computer algebra package such as Magma, [3], to show the result holds. □

Theorem 5.3.3 $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$.

Proof. Suppose $N_G(Y) \cong 3^{\cdot}\mathrm{G}_2(3) : 2$. Then by Lemma 5.3.2 $Z(N_G(Y)) = Y \leq (O_3(P_\delta))'$ for $\delta \in \{\beta, \beta + 2\}$. However, $O_3(P_\delta)' = W'_\delta$ and Lemma 5.3.1 implies that $Y \nleq W'_\delta$, a contradiction. Hence $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$ as required. □

Chapter 6

Proof of Theorem B

This chapter contains results which prove Theorem B. For this chapter we assume that G satisfies Hypothesis A and that in addition, G contains Y, the subgroup of order 3 in G such that $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$ in the conclusions of Theorem A. Therefore, we may use many of the results about the properties of Y from Chapter 5. In particular, we note that $Y \leq C_{Q_\beta}(t_\beta)$ since t_β centralizes Y and $Y \leq Q_\beta$. Similarly, $Z_\beta \leq C_{Q_\beta}(t_\beta)$. As before we have that $\mathcal{F}_3 = \mathcal{F}_3(G_\alpha, G_\beta, G_{\alpha\beta})$ is an amalgam of type F_3 such that G is a faithful completion of $\mathcal{F}_3$ with $N_G(Z_\beta) = G_\beta$. First we prove a number of further results about the structure of $\mathcal{F}_3$ and its associated coset graph. We then consider a section of this coset graph that is fixed by a carefully chosen involution and prove a number of results which we require later in this chapter to prove Theorem B.

6.1 Further Subgroup Structure

Let $\gamma \in \Gamma$ and t_γ be the unique involution described in Remark 5.1.2. We are able to use the results from Chapter 5 about Γ and the structure of Y. The results in this section concern the centralizer of the involution t_γ in various subgroups of G.

Lemma 6.1.1 *Suppose that $\gamma \in \Gamma_\alpha$. Then $C_{G_\gamma}(t_\gamma) \sim 3^3.\mathrm{GL}_2(3)$. In particular, $O_3(C_{G_\gamma}(t_\gamma))$ is elementary abelian.*

Proof. By the choice of t_γ we see that t_γ inverts Z_γ, which is a natural $\mathrm{SL}_2(3)$-module. Therefore t_γ inverts Q_γ/U_γ since the composition factors of Q_γ/U_γ are natural $\mathrm{SL}_2(3)$-modules. So t_γ centralizes at most U_γ/Z_γ on Q_γ. Since $Y \leq Q_\gamma$ and by Lemma 5.1.12, Y is centralized by t_γ, we have that $C_{Q_\gamma}(t_\gamma)$ is non-trivial. Therefore, since U_γ/Z_γ is an $\Omega_3(3)$-module and so is irreducible, we have that $C_{Q_\gamma}(t_\gamma) = C_{U_\gamma}(t_\gamma)$ has order 3^3. Since U_γ is elementary abelian, so is $C_{Q_\gamma}(t_\gamma)$ and hence $O_3(C_{G_\gamma}(t_\gamma)) = C_{Q_\gamma}(t_\gamma) = 3^3$ is elementary abelian. Now $\langle t_\gamma \rangle Q_\gamma \trianglelefteq G_\gamma$ and $\langle t_\gamma \rangle \in \mathrm{Syl}_2(\langle t_\gamma \rangle Q_\gamma)$. So, by the Frattini Lemma, $G_\gamma = N_{G_\gamma}(\langle t_\gamma \rangle)\langle t_\gamma \rangle Q_\gamma = C_{G_\gamma}(t_\gamma)Q_\gamma$. Thus,

$$G_\gamma/Q_\gamma = C_{G_\gamma}(t_\gamma)Q_\gamma/Q_\gamma \cong C_{G_\gamma}(t_\gamma)/(C_{G_\gamma}(t_\gamma) \cap Q_\gamma) = C_{G_\gamma}(t_\gamma)/C_{Q_\gamma}(t_\gamma),$$

and therefore $C_{G_\gamma}(t_\gamma)/C_{Q_\gamma}(t_\gamma) \cong \mathrm{GL}_2(3)$. Hence $C_{G_\alpha}(t_\alpha) \sim 3^3.\mathrm{GL}_2(3)$. $\square$

Lemma 6.1.2 *Suppose $\gamma \in \Gamma_\alpha$. Then $C_{G_\gamma}(t_\gamma)/C_{U_\gamma}(t_\gamma)\langle t_\gamma \rangle \cong \mathrm{Sym}(4)$ and it acts faithfully on $C_{U_\gamma}(t_\gamma)$.*

Proof. By the proof of Lemma 6.1.1, we see that $C_{G_\gamma}(t_\gamma)/C_{U_\gamma}(t_\gamma) = C_{G_\gamma}(t_\gamma)/C_{Q_\gamma}(t_\gamma) \cong \mathrm{GL}_2(3)$. Since $\langle t_\gamma \rangle Q_\gamma = Z(G_\gamma/Q_\gamma) \cong Z(\mathrm{GL}_2(3))$ by definition, we have that t_γ is contained in a group isomorphic to $Z(\mathrm{GL}_2(3))$ and so, $C_{G_\gamma}(t_\gamma)/C_{U_\gamma}(t_\gamma)\langle t_\gamma \rangle \cong \mathrm{GL}_2(3)/\langle t_\gamma \rangle \cong \mathrm{PGL}_2(3) \cong \mathrm{Sym}(4)$. We see that $C_{U_\gamma}(t_\gamma)$ is an orthogonal $C_{G_\gamma}(t_\gamma)/C_{U_\gamma}(t_\gamma)\langle t_\gamma \rangle$-module as $C_{U_\gamma}(t_\gamma)Z_\gamma = U_\gamma$ and hence $C_{G_\gamma}(t_\gamma)/C_{U_\gamma}(t_\gamma)\langle t_\gamma \rangle$ acts faithfully on $C_{U_\gamma}(t_\gamma)$. $\square$

Lemma 6.1.3 *Suppose $\gamma \in \Gamma_\beta$. Then $C_{G_\gamma}(t_\gamma) \sim (3^{1+2} \times \mathrm{Q}_8).3.2$. In particular,*

$$C_{G_\gamma}(t_\gamma)/C_{Q_\gamma}(t_\gamma) \cong \mathrm{GL}_2(3)$$

and so $C_{G_\gamma}(t_\gamma) \sim 3^{1+2}.\mathrm{GL}_2(3)$.

Proof. Let $M_\gamma = O_{3,2}(L_\gamma)$. Let $N_\gamma \in \mathrm{Syl}_2(M_\gamma)$ such that $\langle t_\gamma \rangle \leq N_\gamma$. Then $Q_\gamma N_\gamma = M_\gamma \trianglelefteq L_\gamma$ and $|L_\gamma : M_\gamma| = 3$. Therefore $|G_\gamma : M_\gamma| = 2.3$. By the proof of Lemma

5.1.8, $N_\gamma \cong \mathrm{Q}_8$ and $C_{M_\gamma}(t_\gamma) = C_{Q_\gamma}(t_\gamma)N_\gamma$. So by Lemma 5.1.7, $C_{M_\gamma}(t_\gamma) \cong 3^{1+2} \times \mathrm{Q}_8$. Since $M_\gamma \trianglelefteq L_\gamma \trianglelefteq G_\gamma$ and $N_\gamma \in \mathrm{Syl}_2(M_\gamma)$, by the Frattini Lemma we have that $G_\gamma = N_{G_\gamma}(N_\gamma)M_\gamma$. However, since $Z(N_\gamma) = \langle t_\gamma \rangle$, we have that $N_{G_\gamma}(N_\gamma) \leq C_{G_\gamma}(t_\gamma)$. Therefore $G_\gamma = C_{G_\gamma}(t_\gamma)M_\gamma$. So $C_{G_\gamma}(t_\gamma) = C_{C_{G_\gamma}(t_\gamma)M_\gamma}(t_\gamma) = C_{G_\gamma}(t_\gamma)C_{M_\gamma}(t_\gamma)$, and hence $C_{M_\gamma}(t_\gamma) < C_{G_\gamma}(t_\gamma)$. Therefore $C_{G_\gamma}(t_\gamma) \sim (3^{1+2} \times \mathrm{Q}_8).3.2$. So, since $M_\gamma = N_\gamma Q_\gamma$ we have that $G_\gamma = C_{G_\gamma}(t_\gamma)Q_\gamma$. Therefore, similarly to in the proof of Lemma 6.1.1,

$$G_\gamma/Q_\gamma = C_{G_\gamma}(t_\gamma)Q_\gamma/Q_\gamma \cong C_{G_\gamma}(t_\gamma)/(C_{G_\gamma}(t_\gamma) \cap Q_\gamma) = C_{G_\gamma}(t_\gamma)/C_{Q_\gamma}(t_\gamma),$$

and so $C_{G_\gamma}(t_\gamma)/C_{Q_\gamma}(t_\gamma) \cong \mathrm{GL}_2(3)$. So $C_{G_\gamma}(t_\gamma) \sim 3^{1+2}.\mathrm{GL}_2(3)$. □

We consider a section of the subgraph of Γ that is fixed by t_β. Since by Lemma 5.1.1 *(iii)*, $t_\beta Q_\beta \in Z(L_\beta/Q_\beta)$, and $Q_\beta \leq L_{\gamma\beta} \leq L_\beta$ for $\gamma \in \Gamma(\beta)$, we see that $t_\beta Q_\beta \in Z(L_{\gamma\beta}/Q_\beta)$. So t_β fixes all $\gamma \in \Gamma(\beta)$. Similarly, since $t_\beta = t_{\beta-3}$ by Lemma 5.1.1, *(iv)*, t_β fixes all $\gamma \in \Gamma(\beta - 3)$. Now let $\delta \in \Gamma(\alpha)\backslash\Theta$. Since $L_{\alpha\delta} \nleq L_\beta$, we have that $t_\beta \notin L_{\alpha\delta}$ and so t_β does not fix δ. Similarly, t_β does not fix the neighbours of $\beta - 2$, which are not in Θ. Hence, we can consider the section of the subgraph of Γ that is fixed by t_β in Figure 6.1.

We note that Lemma 5.1.1 *(iii)*, implies that if $\gamma \in \Gamma_\alpha$, then t_γ inverts Z_γ and that if $t_\gamma \in \Gamma_\beta$, then $\langle t_\gamma \rangle = C_T(Z_\gamma)$. Hence, we have that $t_\beta = t_{\beta-3} = t_{\beta-6}$ centralizes both Z_β and $Z_{\beta-6}$ and inverts $Z_{\beta-3}$.

We introduce some further notation.

Definition 6.1.4 For $X \leq G$, let $\widetilde{X} = X \cap C_G(t_\beta) = C_X(t_\beta)$.

We first prove an elementary result that will be useful in this section.

Lemma 6.1.5 *Let $H \leq K$ be subgroups of G. Then $N_{\widetilde{K}}(H) = \widetilde{N_K(H)}$ and $C_{\widetilde{K}}(H) = \widetilde{C_K(H)}$.*

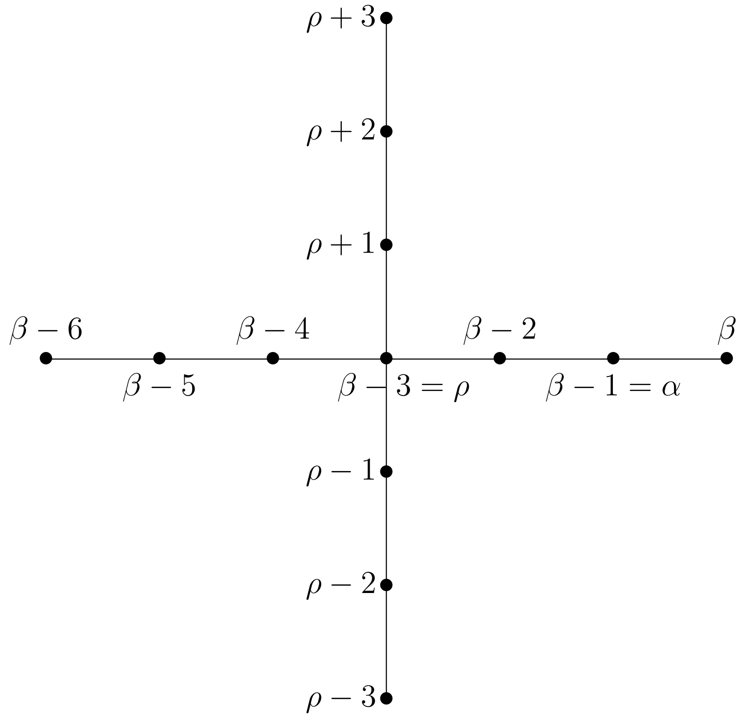

Figure 6.1: Section of the subgraph of Γ fixed by t_β

Proof. We have,

$$\begin{aligned} N_{\widetilde{K}}(H) &= \{k \in \widetilde{K} \mid k^{-1}Hk = H\} \\ &= \{k \in K \mid k^{-1}t_\beta k = t_\beta \text{ and } k^{-1}Hk = H\} \\ &= \{k \in N_K(H) \mid k^{-1}t_\beta k = t_\beta\} \\ &= \widetilde{N_K(H)}. \end{aligned}$$

Similarly, $C_{\widetilde{K}}(H) = \widetilde{C_K(H)}$. □

We recall that $t_\beta \in C_G(Y) \cong 3 \times \mathrm{G}_2(3)$. This leads us to the following result.

Lemma 6.1.6 *We have that* $C_G(Y) \cap \widetilde{G} = \widetilde{C_G(Y)} \sim 3 \times 2_+^{1+4}.(3 \times 3).2$ *and* $N_G(Y) \cap \widetilde{G} = \widetilde{N_G(Y)} \sim (3 \times 2_+^{1+4}.(3 \times 3).2).2$. *In particular, the Sylow* 3*-subgroups of* $\widetilde{C_G(Y)}$ *are elementary abelian of order* 3^3.

Proof. By Lemma 6.1.5, $C_G(Y) \cap \widetilde{G} = C_{\widetilde{G}}(Y) = \widetilde{C_G(Y)}$ and $N_G(Y) \cap \widetilde{G} = N_{\widetilde{G}}(Y) = \widetilde{N_G(Y)}$. Since $\mathrm{G}_2(3)$ contains a unique conjugacy class of involutions, [8, page 61] shows that $C_G(Y) \cap \widetilde{G}$ has shape $3 \times 2^{1+4}_{+}.(3 \times 3).2$. The shape of $N_G(Y) \cap \widetilde{G}$ follows immediately. $\square$

The following lemma will be needed to prove Theorem 6.4.12.

Lemma 6.1.7 *$\widetilde{G}$ contains a subgroup isomorphic to* $2 \times \mathrm{L}_2(8) : 3$.

Proof. By Lemma 5.1.16 (*iii*), there exists an involution in G which is G-conjugate to t_β and inverts Y. Let s be this involution. Hence we have that $C_G(s) \cong C_G(t_\beta) = \widetilde{G}$ and therefore $\widetilde{G} \geq N_G(Y) \cap C_G(s) \cong 2 \times \mathrm{L}_2(8) : 3$, see [8, page 61]. $\square$

We prove a number of results about $\widetilde{S_{\alpha\beta}}$.

Lemma 6.1.8 $C_{\widetilde{S_{\alpha\beta}}}(Y) \in \mathrm{Syl}_3(\widetilde{C_G(Y)})$*. In particular,* $C_{\widetilde{S_{\alpha\beta}}}(Y)$ *is elementary abelian of order* 3^3.

Proof. By Lemma 5.1.17, $W_{\beta-2}W_\beta \in \mathrm{Syl}_3(C_G(Y))$ and so $S_{\alpha\beta} \cap C_G(Y) = W_{\beta-2}W_\beta$. Therefore, $C_{\widetilde{S_{\alpha\beta}}}(Y) = \widetilde{S_{\alpha\beta}} \cap C_G(Y) = \widetilde{W_{\beta-2}W_\beta}$. Therefore, as $\widetilde{W_{\beta-2}W_\beta} \in \mathrm{Syl}_3(C_{\widetilde{G}}(Y))$, we have that $C_{\widetilde{S_{\alpha\beta}}}(Y) \in \mathrm{Syl}_3(C_{\widetilde{G}}(Y))$. Therefore, by Lemma 6.1.6, $C_{\widetilde{S_{\alpha\beta}}}(Y)$ is elementary abelian of order 3^3 as required. $\square$

Lemma 6.1.9 *(i)* $[\widetilde{Q_\beta}, Y] \neq 1$.

(ii) $[\widetilde{Q_\beta}, Z_{\beta-6}] \neq 1$.

Proof. (*i*) This follows immediately from Lemma 6.1.8 since $\widetilde{Q_\beta} \nleq C_{\widetilde{S_{\alpha\beta}}}(Y)$ as it is an extra-special group by Lemma 5.1.7.

(*ii*) Suppose that $[\widetilde{Q_\beta}, Z_{\beta-6}] = 1$. Then $\widetilde{Q_\beta} \leq C_G(Z_{\beta-6}) \leq N_G(Z_{\beta-6}) = G_{\beta-6}$. Hence, $|\widetilde{Q_\beta}Q_{\beta-6}/Q_{\beta-6}| \leq |G_{\beta-6}/Q_{\beta-6}|_3 = |\mathrm{GL}_2(3)|_3 = 3$ and so $\widetilde{Q_\beta}Q_{\beta-6}/Q_{\beta-6}$ is elementary abelian. Since $\widetilde{Q_\beta}Q_{\beta-6}/Q_{\beta-6} \cong \widetilde{Q_\beta}/\widetilde{Q_\beta} \cap Q_{\beta-6}$ this implies that $\widetilde{Q_\beta}/\widetilde{Q_\beta} \cap Q_{\beta-6}$

is elementary abelian. Therefore $\Phi(\widetilde{Q_\beta}) \leq \widetilde{Q_\beta} \cap Q_{\beta-6}$. Since by Lemma 5.1.7, $\widetilde{Q_\beta}$ is extra-special, $\Phi(\widetilde{Q_\beta}) = Z_\beta$. Therefore $Z_\beta \leq \widetilde{Q_\beta} \cap Q_{\beta-6} \leq Q_{\beta-6}$. This is a contradiction since the critical distance is 5 and G acts 7-arc transitively on Γ. $\square$

Lemma 6.1.10 *T inverts Z_β and $Z_{\beta-6}$.*

Proof. Since $Z_{\beta-6}$ is G-conjugate to Z_β, it suffices to show that T inverts Z_β. We have that Z_α is elementary abelian of order 3^2. Therefore, since T is a Klein four-group which acts faithfully on Z_α, Proposition 1.3.5 implies that T inverts Z_α. Since $Z_\beta \leq Z_\alpha$, it remains to show that Z_β is T-invariant. If Z_β was not T-invariant, then any non-trivial element of T would not centralize Z_β. Clearly t_β centralizes Z_β by definition and hence Z_β is T-invariant. $\square$

Lemma 6.1.11 *T normalizes $\widetilde{S_{\alpha\beta}}$.*

Proof. Since $S_{\alpha\beta} \trianglelefteq G_{\alpha\beta}$ we have that T normalizes $S_{\alpha\beta}$. Also $T \leq \widetilde{G}$ and therefore T normalizes $S_{\alpha\beta} \cap \widetilde{G} = \widetilde{S_{\alpha\beta}}$. $\square$

Lemma 6.1.12 *$Z(\widetilde{S_{\alpha\beta}}) = Z_\beta$.*

Proof. By Lemma 6.1.11, $\widetilde{S_{\alpha\beta}}$ is T-invariant. Hence, since $Z(\widetilde{S_{\alpha\beta}}) \trianglelefteq \widetilde{S_{\alpha\beta}}$, so is $Z(\widetilde{S_{\alpha\beta}})$. As $[\widetilde{Q_\beta}, Y] \neq 1$ by Lemma 6.1.9, we have that $Z(\widetilde{S_{\alpha\beta}}) \geq Z(\widetilde{S_{\alpha\beta}}) \cap \widetilde{Q_\beta} = Z(\widetilde{Q_\beta}) = Z_\beta$ since $\widetilde{Q_\beta}$ is an extra-special group by Lemma 5.1.7. Also, $Z_\beta < C_{\widetilde{S_{\alpha\beta}}}(Y)$. So, suppose that $|Z(S_{\alpha\beta})| = 9$. By Lemma 6.1.8, $C_{\widetilde{S_{\alpha\beta}}}(Y)$ is elementary abelian of order 3^3, and hence $C_{\widetilde{S_{\alpha\beta}}}(Y)/Z_\beta$ is elementary abelian of order 3^2. We can consider T as the subgroup of $\mathrm{GL}_2(3)$ generated by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, by Proposition 1.3.5, $C_{\widetilde{S_{\alpha\beta}}}(Y)/Z_\beta$ contains two subgroups of order 3 that are invariant under the action of T. Since Y and $Z_{\beta-6}$ are invariant under this action by Lemma 6.1.10, we see that the subgroups of $C_{\widetilde{S_{\alpha\beta}}}(Y)/Z_\beta$ that are invariant under the action of T are YZ_β/Z_β and $Z_{\beta-6}Z_\beta/Z_\beta$. Hence, either $Z(\widetilde{S_{\alpha\beta}}) = YZ_\beta$, or $Z(\widetilde{S_{\alpha\beta}}) = Z_{\beta-6}Z_\beta$. This is a contradiction since $[\widetilde{Q_\beta}, Y] \neq 1$ and $[\widetilde{Q_\beta}, Z_{\beta-6}] \neq 1$ by Lemma 6.1.9. Hence $|Z(\widetilde{S_{\alpha\beta}})| = 3$ and $Z(\widetilde{S_{\alpha\beta}}) = Z_\beta$. $\square$

The next lemma uses our global hypothesis that $G_\beta = N_G(Z_\beta)$.

Lemma 6.1.13 $\widetilde{S_{\alpha\beta}} \in \mathrm{Syl}_3(\widetilde{G})$.

Proof. Let $E_1 \geq \widetilde{S_{\alpha\beta}}$ such that $E_1 \in \mathrm{Syl}_3(\widetilde{G})$. Then $N_{E_1}(\widetilde{S_{\alpha\beta}}) \geq \widetilde{S_{\alpha\beta}}$ since $\widetilde{S_{\alpha\beta}}$ is 3-group. As $Z(\widetilde{S_{\alpha\beta}})$ is a characteristic subgroup of $\widetilde{S_{\alpha\beta}}$ we have that $Z(\widetilde{S_{\alpha\beta}}) \trianglelefteq N_{E_1}(\widetilde{S_{\alpha\beta}}) \leq E_1$. Therefore,

$$\begin{aligned} N_{E_1}(\widetilde{S_{\alpha\beta}}) &\trianglelefteq N_{E_1}(Z(S_{\alpha\beta})) \\ &= N_{E_1}(Z_\beta) && \text{by Lemma 6.1.12} \\ &\leq G_\beta. \end{aligned}$$

So $N_{E_1}(\widetilde{S_{\alpha\beta}}) = \widetilde{S_{\alpha\beta}}$ and thus $E_1 = \widetilde{S_{\alpha\beta}}$. Hence $\widetilde{S_{\alpha\beta}} \in \mathrm{Syl}_3(\widetilde{G})$. $\square$

Lemma 6.1.14 $\widetilde{S_{\alpha\beta}}$ *is isomorphic to a Sylow* 3*-subgroup of* $\mathrm{Alt}(9)$.

Proof. Let $D = Z_{\beta-6}Y$. Then $D \leq \widetilde{S_{\alpha\beta}}$ and $|\widetilde{S_{\alpha\beta}} : D| = 3^2$. Let ϕ be the action of $\mathrm{S}_{\alpha\beta}$ on the right cosets of D. So $\phi : \widetilde{S_{\alpha\beta}} \to R$ where $R \leq \mathrm{Sym}(9)$. By the First Isomorphism Theorem, $\widetilde{S_{\alpha\beta}}/\ker\phi \cong R$. Suppose that $\ker\phi \neq 1$. Since $\ker\phi \leq \widetilde{S_{\alpha\beta}}$, by Lemma 1.1.12, $Z(\widetilde{S_{\alpha\beta}}) \cap \ker\phi \neq 1$. So $Z(\widetilde{S_{\alpha\beta}}) \leq \ker\phi$ since $Z(\widetilde{S_{\alpha\beta}}) = Z_\beta$ by Lemma 6.1.12. However, $\ker\phi \leq D$ and since $D \cap Z_\beta = 1$ this leads to a contradiction. Hence $\ker\phi = 1$. Therefore $\widetilde{S_{\alpha\beta}}$ is isomorphic to a subgroup of $\mathrm{Sym}(9)$, and hence of $\mathrm{Alt}(9)$ of order 3^4. Since the Sylow 3-subgroups of $\mathrm{Alt}(9)$ have order 3^4, this implies that $\widetilde{S_{\alpha\beta}}$ is isomorphic to a Sylow 3-subgroup of $\mathrm{Alt}(9)$. $\square$

6.2 The Subgroup J

Let $J = J(\widetilde{S_{\alpha\beta}})$ be the Thompson subgroup of $\widetilde{S_{\alpha\beta}}$.

Lemma 6.2.1 $C_{\widetilde{S_{\alpha\beta}}}(Y) = \widetilde{Q_{\beta-3}} = \widetilde{U_{\beta-3}} = \langle Y, Z_\beta, Z_{\beta-6}\rangle = J.$

Proof. By Lemma 6.1.14, we may consider $\widetilde{S_{\alpha\beta}}$ as a Sylow 3-subgroup of $\mathrm{Alt}(9)$. Hence J is elementary abelian of order 3^3. Therefore $J = C_{\widetilde{S_{\alpha\beta}}}(Y)$ by orders. Clearly $Y, Z_\beta, Z_{\beta-6} \in \mathcal{A}(\widetilde{S_{\alpha\beta}})$ and hence $J = \langle Y, Z_\beta, Z_{\beta-6}\rangle$. Since $\beta - 3 \in \Theta\backslash\Theta_\beta$, by the proof of Lemma 6.1.1, we have that $\widetilde{Q_{\beta-3}} = \widetilde{U_{\beta-3}}$ has order 3^3 and is elementary abelian. Since $U_{\beta-3} \leq S_{\alpha\beta}$ this implies that $\widetilde{U_{\beta-3}} \leq J$. Hence by orders, equality holds. $\square$

Lemma 6.2.2 $C_{\widetilde{G}}(J) = J\langle t_\beta\rangle$.

Proof. Clearly $J \leq C_{\widetilde{G}}(J)$ since it is elementary abelian. Since t_β centralizes Y, Z_β and $Z_{\beta-6}$, we also have that t_β centralizes $\langle Y, Z_\beta, Z_{\beta-6}\rangle = J$. Hence $J\langle t_\beta\rangle \leq C_{\widetilde{G}}(J)$.

Since $\widetilde{S_{\alpha\beta}} \nleq C_{\widetilde{G}}(J)$ we have that $|C_{\widetilde{G}}(J)|_3 \leq 3^3$. However, since $J \leq C_{\widetilde{G}}(J)$ and $|J| = 3^3$, this implies that $J \in \mathrm{Syl}_3(C_{\widetilde{G}}(J))$.

As $Z_\beta \leq J$, we have that $C_{\widetilde{G}}(J) \leq C_{\widetilde{G}}(Z_\beta) \leq C_{\widetilde{G}}(Z_\beta) = \widetilde{C_G(Z_\beta)} = \widetilde{G_\beta}$. By Lemma 6.1.3, $\widetilde{G_\beta} \sim (3^{1+2} \times \mathrm{Q}_8).3.2$ and so $\widetilde{G_\beta}/\widetilde{Q_\beta} \cong \mathrm{GL}_2(3)$. Therefore, since t_β is the unique element of T which centralizes Z_β and T can be thought of as a subgroup of $\mathrm{GL}_2(3)$, we see that $C_{\widetilde{G}}(J) = C_{\widetilde{G_\beta}}(J) = J\langle t_\beta\rangle$. $\square$

We show that J is a faithful $\mathrm{GF}(3)\,\mathrm{Sym}(4)$-module, enabling us to use some of the results from Section 3.1.

Lemma 6.2.3 *J is a faithful* $\mathrm{GF}(3)\,\mathrm{Sym}(4)$*-module.*

Proof. By Lemma 6.1.2, $X = \widetilde{G_{\beta-3}}/\widetilde{U_{\beta-3}}\langle t_\beta\rangle \cong \mathrm{Sym}(4)$ and X acts faithfully on $\widetilde{U_{\beta-3}} = J$. Hence J is a faithful $\mathrm{GF}(3)\,\mathrm{Sym}(4)$-module. $\square$

Lemma 6.2.4 $N_{\widetilde{G}}(J)/C_{\widetilde{G}}(J) \cong \mathrm{Sym}(4)$. *In particular,* $N_{\widetilde{G}}(J) = \widetilde{G_{\beta-3}}$.

Proof. Since $J \cong 3^3$, we have that $\mathrm{Aut}(J) \cong \mathrm{GL}_3(3)$ by [13, Theorem 1.3.2]. Therefore $N_{\widetilde{G}}(J)/C_{\widetilde{G}}(J)$ is isomorphic to a subgroup of $\mathrm{GL}_3(3)$. Also, $N_{\widetilde{G}}(J)/C_{\widetilde{G}}(J)$ contains a subgroup isomorphic to $\mathrm{Sym}(4)$ since $N_{\widetilde{G}}(J) \geq \widetilde{G_{\beta-3}} \cong 2.3^3.\mathrm{Sym}(4)$ by Lemma 6.1.1. Therefore using [8, page 13] we see that $N_{\widetilde{G}}(J)/C_{\widetilde{G}}(J)$ is isomorphic to either $\mathrm{Sym}(4)$ or $2 \times \mathrm{Sym}(4)$. Suppose that $N_{\widetilde{G}}(J)/C_{\widetilde{G}}(J) \cong 2 \times \mathrm{Sym}(4)$. Then by Lemma 6.1.12, $Z(\widetilde{S_{\alpha\beta}}) = Z_\alpha$ and hence $N_{\widetilde{G}}(\widetilde{S_{\alpha\beta}}) = \widetilde{G_\beta} \cap N_{\widetilde{G}}(\widetilde{S_{\alpha\beta}}) = N_{\widetilde{G_\beta}}(\widetilde{S_{\alpha\beta}}) = \widetilde{S_{\alpha\beta}}T$. Since $\widetilde{G_{\beta-3}} \geq \widetilde{S_{\alpha\beta}}T$, this implies that $\widetilde{G_{\beta-3}} \trianglelefteq N_{\widetilde{G}}(J)$. Therefore, the Frattini Lemma implies that $N_{\widetilde{G}}(J) = N_{N_{\widetilde{G}}(J)}(\widetilde{S_{\alpha\beta}})\widetilde{G_{\beta-3}}$ and hence $N_{N_{\widetilde{G}}(J)}(\widetilde{S_{\alpha\beta}}) \not\leq \widetilde{G_{\beta-3}}$. This is a contradiction and hence $N_{\widetilde{G}}(Y)/C_{\widetilde{G}}(J) \cong \mathrm{Sym}(4)$. Therefore, by Lemma 6.2.2 we have that $|N_{\widetilde{G}}(J)| = |C_{\widetilde{G}}(J)||\mathrm{Sym}(4)| = |\widetilde{G_{\beta-3}}|$ and hence $N_{\widetilde{G}}(J) = \widetilde{G_{\beta-3}}$. $\square$

Lemma 6.2.5 $N_{\widetilde{G}/\langle t_\beta \rangle}(J) \cong 3^3 : \mathrm{Sym}(4)^+$.

Proof. By Lemma 6.2.4,

$$F = N_{\widetilde{G}/\langle t_\beta \rangle}(J) \cong H \in \{3^3 : \mathrm{Sym}(4)^+, 3^3 : \mathrm{Sym}(4)^-\}.$$

We have that $O_3(F) = J \cong 3^3$ and so $O_3(F) \cong U_\alpha/Z_\alpha$. From the proof of Lemma 5.1.16, we see that $|C_{U_\alpha/Z_\alpha}(t_\beta)| = 3$ and therefore $|C_F(t_\beta)| = 2^2.3$. Since $t_\beta \notin F'$, by Theorem 3.1.6, $N_F(t_\beta) = N_{\widetilde{G}/\langle t_\beta \rangle}(J) \cong 3^3 : \mathrm{Sym}(4)^+$. $\square$

Lemma 6.2.6 $\widetilde{G_{\beta-3}}$ *has orbits of lengths* 3, 4 *and* 6 *on the cyclic subgroups of* J. *These orbits are not fused in* $\widetilde{G}$.

Proof. Since J is a faithful $\mathrm{GF}(3)\,\mathrm{Sym}(4)$-module by Lemma 6.2.3 the first part of this lemma follows from Lemma 3.1.2. These orbits do not fuse in $\widetilde{G}$ by Lemma 1.1.11 since $J = J(\widetilde{S_{\alpha\beta}})$ where $\widetilde{S_{\alpha\beta}} \in \mathrm{Syl}_3(\widetilde{G})$ by Lemma 6.1.13. $\square$

We recall $\rho+3$ and $\rho-3$ from Figure 6.1.

Lemma 6.2.7 *Let $\mu \in \{\beta, \beta-6, \rho-3, \rho+3\}$. Then $Z_\mu \leq J$.*

Proof. We have that $[Z_\beta, t_\beta] = 1$ by definition. By Lemma 5.1.1 (iv), if $d(\mu, \pi) = 3$, for $\mu, \pi \in \Theta$, then $t_\mu = t_\pi$. Hence $[Z_{\beta-6}, t_\beta] = [Z_{\rho-3}, t_\beta] = [Z_{\rho+3}, t_\beta] = 1$. Hence $Z_\mu \leq \widetilde{S_{\alpha\beta}}$ for all $\mu \in \{\beta, \beta-6, \rho-3, \rho+3\}$.

Since by Lemma 6.2.1, $J = \widetilde{S_{\alpha\beta}} \cap C_{S_{\alpha\beta}}(Y)$ it remains to show that Z_μ centralizes Y. Let $\mu \in \{\beta, \beta-6, \rho-3, \rho+3\}$. Then $Z_\mu \leq Z_{\mu-1} \leq U_{\mu-1} = W_\mu \cap W_{\mu-2} \leq P_\mu \leq C_G(Y)$ and so $[Z_\mu, Y] = 1$. Therefore, since $Z_\mu \leq S_{\alpha\beta}$, we have that $Z_\mu \leq C_{S_{\alpha\beta}}(Y)$, completing the proof of the lemma. □

Let $X = \langle Z_\mu \mid \mu \in \{\beta, \beta-6, \rho-3, \rho+3\}\rangle$. Then Lemma 6.2.7 implies that $X \leq J$ and hence X is elementary abelian.

Lemma 6.2.8 *$\widetilde{G_{\beta-3}}$ acts transitively on $\{\beta, \beta-6, \rho-3, \rho+3\}$. In particular, $X = J \trianglelefteq \widetilde{G_{\beta-3}}$.*

Proof. Since the path of length 3 from $\beta-3=\rho$ is uniquely determined by the element of $\Gamma(\beta-3)$ it passes through, it suffices to prove that $\widetilde{G_{\beta-3}}$ acts transitively on $\Gamma(\beta-3)$.

By Lemma 1.5.4, $G_{\beta-3} = G_\rho$ acts transitively on $\Gamma(\beta-3) = \Gamma(\rho)$. Since $Q_\rho \leq G_{\rho\gamma}$ for $\gamma \in \Gamma(\rho)$, and t_β stabilizes $\Gamma(\rho)$ point-wise, we see that $t_\beta Q_\rho$ is the point-wise stabilizer of $\Gamma(\rho)$ in G_ρ. Now, $\langle t_\beta\rangle \in \mathrm{Syl}_2(\langle t_\beta\rangle Q_\rho)$ and $\langle t_\beta\rangle Q_\rho \trianglelefteq G_\beta$. Hence, by the Frattini Lemma, $G_\rho = N_{G_\rho}(\langle t_\beta\rangle)\langle t_\beta\rangle Q_\rho$. Since $\widetilde{G_\rho} \leq G_\rho$ we apply Lemma 1.1.17 to $G = G_{\beta-3} = G_\rho$, $\Omega = \Gamma(\beta-3)$, $K = \langle t_\beta\rangle Q_{\beta-3}$ and $H = \widetilde{G_{\beta-3}}$ to see that $\widetilde{G_{\beta-3}}$ acts transitively on $\Gamma(\beta-3)$ and hence on $\{\beta, \beta-6, \rho-3, \rho+3\}$.

Therefore, $X \trianglelefteq \widetilde{G_{\beta-3}}$. Since $X \leq J$ and $\widetilde{G_{\beta-3}} \sim 3^3.\mathrm{GL}_2(3)$ by Lemma 6.1.1, we have that $X = J$. □

Lemma 6.2.9 *(i) Z_β has four conjugates in $\widetilde{G_{\beta-3}}$ and these conjugates form an orbit of $\widetilde{G_{\beta-3}}$ on the cyclic subgroups of order 3 of length 4.*

(ii) Y *has six conjugates in* $\widetilde{G_{\beta-3}}$ *and these conjugates form an orbit of* $\widetilde{G_{\beta-3}}$ *on the cyclic subgroups of order* 3 *of length* 6*.*

Proof. (i) This follows from Lemma 6.2.8.

(ii) By the Orbit-Stabilizer Theorem, $|\widetilde{G_{\beta-3}} : N_{\widetilde{G_{\beta-3}}}(Y)| = |\{Y^{\widetilde{G_{\beta-3}}}\}|$. We have,

$$\begin{aligned} N_{\widetilde{G_{\beta-3}}}(Y) &= N_G(Y) \cap \widetilde{G_{\beta-3}} \\ &= N_G(Y) \cap N_{\widetilde{G}}(J) && \text{by Lemma 6.2.4} \\ &= N_{\widetilde{N_G(Y)}}(J). \end{aligned}$$

By Lemma 6.2.1, $J = \widetilde{U_{\beta-3}}$. Since $\widetilde{U_{\beta-3}} \in \mathrm{Syl}_3(\widetilde{N_G(Y)})$, by Sylow's Theorem, $|\widetilde{N_G(Y)} : N_{\widetilde{N_G(Y)}}(\widetilde{U_{\beta-3}})| = n$ such that $n \equiv 1 \mod 3$ and n divides 2^7. Therefore, $n = 2^2$, 2^4 or 2^6 and hence

$$|N_{\widetilde{N_G(Y)}}(Y)| = |N_{\widetilde{N_G(Y)}}(\widetilde{U_{\beta-3}})| = \frac{2^7 3^3}{n} = \begin{cases} 2.3^3, & \text{if } n = 2^2; \\ 2^3 3^3, & \text{if } n = 2^4; \\ 2^5 3^3, & \text{if } n = 2^6. \end{cases}$$

Thus

$$|\widetilde{G_{\beta-3}} : N_{\widetilde{G_{\beta-3}}}(Y)| = \begin{cases} 2^3 3, & \text{if } n = 2^2; \\ 2.3, & \text{if } n = 2^4; \\ 2^{-1}3, & \text{if } n = 2^6. \end{cases}$$

We recall from Lemma 6.2.6 that $\widetilde{G_{\beta-3}}$ has orbits of lengths 3, 4 and 6 on the cyclic subgroups of J. Also Y is not $\widetilde{G_{\beta-3}}$-conjugate to Z_β. Therefore, by (i) and the above we have that the orbit length of $\widetilde{G_{\beta-3}}$ on Y is 6. Hence Y has six conjugates in $\widetilde{G_{\beta-3}}$ and the result follows. □

Lemma 6.2.10 *Let V be a faithful* $\mathrm{GF}(3)\,\mathrm{Sym}(4)$*-module with basis $\{v_1, v_2, v_3\}$ as in Lemma 3.1.1. Then the subgroups conjugate to Z_β in $\widetilde{G_{\beta-3}}$ correspond to the* 1*-dimensional subspaces of V with representative $\langle v_1 + v_2 + v_3\rangle$. The subgroups conjugate to Y in $\widetilde{G_{\beta-3}}$ correspond to the* 1*-dimensional subspaces of V with representative $\langle v_1 + v_2\rangle$.*

Proof. This follows from Lemmas 3.1.1, 6.2.3 and 6.2.9. $\square$

6.3 Subgroups of J of Order 3^2

We now consider the subgroups of J of index 3. We show that there are three types of them and prove some results about the centralizer of each of them in $\widetilde{G}$.

Again, we require our global hypothesis that $G_\beta = N_G(Z_\beta)$ in the following result.

Lemma 6.3.1 *Suppose that $K \trianglelefteq \widetilde{G}$ such that* 3 *divides $|K|$. Then $J \leq K$ and K has even order.*

Proof. Let K be a proper normal subgroup of $\widetilde{G}$ and suppose that 3 divides $|K|$. By Lemma 6.1.13, $\widetilde{S_{\alpha\beta}} \in \mathrm{Syl}_3(\widetilde{G})$. So $1 < K \cap \widetilde{S_{\alpha\beta}} \in \mathrm{Syl}_3(K)$ and $K \cap \widetilde{S_{\alpha\beta}} \trianglelefteq \widetilde{S_{\alpha\beta}}$. So by Lemma 6.1.12, $Z_\beta = Z(\widetilde{S_{\alpha\beta}}) \leq K$.

By assumption K is normalized by $\widetilde{G}$. Hence K is also normalized by $\widetilde{G_\beta}$ and $\widetilde{G_{\beta-3}}$. Therefore $K \cap \widetilde{G_{\beta-3}} \geq \langle Z_\beta^{\widetilde{G_{\beta-3}}}\rangle = J$. Hence $J \leq K$. Also $K \cap \widetilde{G_\beta} \geq \langle J^{\widetilde{G_\beta}}\rangle = JO_2(\widetilde{G_\beta})$. Since by Lemma 6.1.3, $\widetilde{G_\beta} \sim (3^{1+2} \times \mathrm{Q}_8).3.2$ we have that $O_2(\widetilde{G_\beta}) \cong \mathrm{Q}_8$. Therefore K has even order. $\square$

We introduce some further notation.

Notation 6.3.2 For A a finite group, $B \leq A$ and π a set of primes, let $\text{И}_A(B, \pi)$ denote the set of B-invariant π-subgroups of A and $\text{И}^*_A(B, \pi)$ denote the maximal subgroups by inclusion in this set.

Let $R \in И^*_{\widetilde{G}}(J, 3')$. So, J is a 3-group acting on R, a $3'$-group, and hence by Coprime Action, $R = \langle C_R(j) \mid j \in J^\# \rangle$. However, it is more convenient to express R in terms of the centralizers of subgroups of R of order 3^2 and hence, again by Coprime Action,

$$R = \langle C_R(\langle j_1, j_2 \rangle) \mid \langle j_1, j_2 \rangle \leq J \text{ and } |\langle j_1, j_2 \rangle| = 3^2 \rangle.$$

We follow a method similar to that used in [29, Section 3] and consider $C_{\widetilde{G}}(A)$ for $|J : A| = 3$. We note that the conjugates of Z_β are 3-central subgroups of G.

Lemma 6.3.3 *The subgroups of J of order 3^2 are of the following types.*

Type 1 *These contain one 3-central subgroup of G and three cyclic subgroups H with $C_G(H) \cong 3 \times \mathrm{G}_2(3)$. There are four such subgroups.*

Type 2 *These contain exactly two 3-central subgroups of G. There are six such subgroups.*

Type 3 *These do not contain any 3-central subgroups of G and contain exactly two cyclic subgroups H with $C_G(H) \cong 3 \times \mathrm{G}_2(3)$. There are three such subgroups.*

Proof. The subgroups of J of order 3 which are not $G_{\beta-3} = N_{\widetilde{G}}(J)$-conjugate to Z_β are not G-conjugate to Z_β by Lemma 1.1.11. The result then follows from Lemmas 3.1.1 and 6.2.10. □

Lemma 6.3.4 *Suppose that $A \leq J$ such that $|J : A| = 3$. Let $b \in A$ such that b is contained in a cyclic subgroup H of A such that $C_G(H) \cong 3 \times \mathrm{G}_2(3)$. Then $O_{3'}(C_{\widetilde{G}}(A)) \leq O_{3'}(C_{\widetilde{G}}(b)) \cong 2^{1+4}_+$.*

Proof. Suppose that A and b satisfy the hypothesis. Hence $C_G(b) \cong 3 \times \mathrm{G}_2(3)$. So by Lemma 6.1.6, $\widetilde{C_G(b)} \sim 3 \times 2^{1+4}_+.(3 \times 3) : 2$ and therefore $O_{3'}(\widetilde{C_G(b)}) \cong 2^{1+4}_+$. Clearly, $\widetilde{C_G(A)} \leq \widetilde{C_G(b)}$ as $b \in A$ by assumption. In $\widetilde{C_G(b)}/Y \cong 2^{1+4}_+(3 \times 3) : 2 \leq \mathrm{G}_2(3)$ we have that the quotient group $3^2.2$ inverts the Sylow 3-subgroups of $\widetilde{C_G(b)}/Y$ and therefore it inverts the Sylow 3-subgroup of $\widetilde{C_G(A)}/Y$. Hence $O_{3'}(\widetilde{C_G(A)}) \leq O_{3'}(\widetilde{C_G(b)}) \cong 2_+^{1+4}$. $\square$

Before proving the next result we recall from Definition 5.1.9 that if $\gamma \in \Theta_\beta$, and $(\gamma-2, \gamma-1, \gamma, \gamma+1, \gamma+2)$ is a path of length 4 in Θ, $P_\gamma = \langle W_{\gamma-2}, W_\gamma, W_{\gamma+2}\rangle T$.

We note that we are using the Atlas, [8] notation for the conjugacy classes of $\mathrm{G}_2(3)$ in the following result and Lemma 6.3.6.

Lemma 6.3.5 *Suppose A is a subgroup of $C_G(Y)'$ such that $|A| = 3$ and the non-trivial elements of A are in $C_G(Y)'$-conjugacy class 3A or 3B. Then A is 3-central in G.*

Proof. Let $N = C_G(Y)'$. By Lemma 4.2.2 (*ii*), $W_\beta' = Z_\beta$. Therefore $Z_\beta \leq N$. We have that $C_N(Z_\beta) = P_\beta \cap N$ and so the non-trivial elements of Z_β are in N-class 3A or 3B which we note are fused in $\mathrm{Aut}(\mathrm{G}_2(3))$. Similarly, the non-trivial elements of $Z_{\beta+2}$ are in class 3A or 3B. Therefore, any subgroup $\langle a\rangle$ where a is an element in N-class 3A or 3B is conjugate in N to either Z_β or $Z_{\beta+2}$ and hence is 3-central in G. $\square$

We now fix a basis $\{v_1, v_2, v_3\}$ for V and a subgroup Y that corresponds to the subspace $\langle v_1+v_2\rangle$ and consider the two dimensional subspaces of V that contain Y. We have that $C_G(Y) \cong 3 \times \mathrm{G}_2(3)$.

A_1 : $\langle v_1+v_2, v_2+v_3\rangle$. This contains the 1-dimensional subspaces $\langle v_1+v_2\rangle$, $\langle v_2+v_3\rangle$, $\langle -v_1+v_3\rangle$ and $\langle v_1-v_2+v_3\rangle$ and is of type 1.

A_1^* : $\langle v_1+v_2, v_1+v_3\rangle$. This contains the 1-dimensional subspaces $\langle v_1+v_2\rangle$, $\langle v_1+v_3\rangle$, $\langle -v_2+v_3\rangle$ and $\langle -v_1+v_2+v_3\rangle$ and is of type 1.

A_2 : $\langle v_1+v_2, v_3\rangle$. This contains the 1-dimensional subspaces $\langle v_1+v_2\rangle$, $\langle v_3\rangle$, $\langle v_1+v_2+v_3\rangle$ and $\langle v_1+v_2-v_3\rangle$ and is of type 2.

A_3 : $\langle v_1, v_2\rangle$. This contains the 1-dimensional subspaces $\langle v_1+v_2\rangle$, $\langle v_1\rangle$, $\langle v_2\rangle$ and $\langle -v_1+v_2\rangle$ and is of type 3.

So, a given conjugate of Y is contained in exactly two subgroups of type 1, one of type 2 and one of type 3. We let s be an involution in $N_{\widetilde{G_\beta}}(J)$ which centralizes Y and commutes with t_β such that s interchanges the subspaces A_1 and A_1^*. Then s acts as the involution $(12) \in \mathrm{Sym}(4)$ on the subspaces $\langle v_1 \rangle$, $\langle v_2 \rangle$ and $\langle v_3 \rangle$. We note that $s \in C_{N_{\widetilde{G}}(J)}(Y)$.

Lemma 6.3.6 *Let A be a subgroup of J of type 3. Then $C_{O_{3'}(C_{\widetilde{G}}(Y))}(A) = \langle t_\beta \rangle$.*

Proof. Let $N = C_G(Y)' = \mathrm{G}_2(3)$. Then $O_{3'}(\widetilde{N}) = O_{3'}(C_{\widetilde{G}}(Y)) = 2_+^{1+4}$. So by Lemma 1.2.7, $C_{O_{3'}(\tilde{N})}(A \cap N)$ has order 2 or 8. Suppose that $|C_{O_{3'}(\tilde{N})}(A \cap N)| = 8$. Then $C_N(A \cap N)$ must contain a subgroup of order 8. So we consider the centralizers of elements of order 3 in $N = \mathrm{G}_2(3)$ using [8, page 61]. Therefore, either:

(*i*) $C_N(A \cap N) \cong (3_+^{1+2} \times 3^2) : 2.\,\mathrm{Sym}(4)$;

(*ii*) $|C_N(A \cap N)| = 729$; or

(*iii*) $|C_N(A \cap N)| = 162$.

Since 729 and 162 are not divisible by 8 we see that we only need consider case (*i*). In this case the non-trivial elements of $A \cap N$ are in N-conjugacy class 3A or 3B, and hence 3-central in G by Lemma 6.3.5. This is a contradiction since subgroups of type 3 do not contain any central subgroups of order 3, see Lemma 6.3.3. Hence $C_{O_{3'}(\tilde{N})}(A \cap N)$ has order 2. We note that $\langle t_\beta \rangle \leq C_{O_{3'}(C_{\widetilde{G}}(Y))}(A)$ and so,

$$\langle t_\beta \rangle = C_{O_{3'}(\tilde{N})}(A \cap N) \geq C_{O_{3'}(\tilde{N})}(A) = C_{O_{3'}(C_{\widetilde{G}}(Y))}(A) \geq \langle t_\beta \rangle.$$

So $C_{O_{3'}(C_{\widetilde{G}}(Y))}(A) = \langle t_\beta \rangle$ as required. □

Lemma 6.3.7 $C_{O_{3'}(C_{\widetilde{G}}(Y))}(A_1) \cong C_{O_{3'}(C_{\widetilde{G}}(Y))}(A_1^*)$.

Proof. We have that A_1 and A_1^* are conjugated by elements of $O_{3'}(C_{N_{\widetilde{G}}(J)}(Y))$ since the involution $s \in O_{3'}(C_{N_{\widetilde{G}}(J)}(Y))$ interchanges the subspaces $\langle v_1 \rangle$ and $\langle v_2 \rangle$. Hence the lemma follows. $\square$

Lemma 6.3.8 *Let A_1, A_1^*, A_2 and A_3 be the 2-dimensional subspaces of V which contain the 1-dimensional subspace $\langle v_1 + v_2 \rangle$. Then:*

(i) $O_{3'}(C_{\widetilde{G}}(A_1)) \cong O_{3'}(C_{\widetilde{G}}(A_1^*)) \cong \mathrm{Q}_8$;

(ii) $O_{3'}(C_{\widetilde{G}}(A_2)) \cong \langle t_\beta \rangle$; *and*

(iii) $O_{3'}(C_{\widetilde{G}}(A_3)) \cong \langle t_\beta \rangle$.

In addition, $O_{3'}(C_{\widetilde{G}}(A_1)) \neq O_{3'}(C_{\widetilde{G}}(A_1^*))$ and they commute. In particular, $\text{И}^*_{C_{\widetilde{G}}(A_i)}(J, 3') = \{O_{3'}(C_{\widetilde{G}}(A_i))\}$ for $i = 1, 2, 3$.

Proof. By Lemma 6.3.4, it suffices to consider the centralizers of these subgroups of order 3^2 in $F \cong 2_+^{1+4}$. So let A_1, A_1^*, A_2 and A_3 be as defined above. By Lemma 6.3.6, $C_F(A_3) = C_{O_{3'}(C_{\widetilde{G}}(Y))}(A_3) = \langle t_\beta \rangle$. We also have that, $C_F(A_1) \cong C_F(A_1^*)$ by Lemma 6.3.7. Since $(|F|, |A|) = 1$ for all $A \in \{A_1, A_1^*, A_2, A_3\}$, Coprime Action implies that

$$F \cong \langle C_F(A_1), C_F(A_1^*), C_F(A_2), C_F(A_3) \rangle.$$

Suppose that $C_F(A_1) \cong \langle t_\beta \rangle$. Then $C_F(A_1) \cong C_F(A_1^*) \cong C_F(A_3)$. Hence, $F \cong C_F(A_2) \cong \mathrm{Q}_8$ or $\langle t_\beta \rangle$. In either case we have a contradiction and so $C_F(A_1) \cong \mathrm{Q}_8$. Therefore $C_F(A_1^*) \cong \mathrm{Q}_8$ and this completes the proof of (i).

Now suppose that $C_F(A_1) = C_F(A_1^*)$. Then since $A_1 A_1^* = J$, we have that $C_F(A_1) = C_F(J)$. We know that $C_F(J) \leq C_F(A_3) = \langle t_\beta \rangle$ since $A_3 \leq J$. This is a contradiction as $C_F(A_1) \cong \mathrm{Q}_8$. Hence $C_F(A_1) \neq C_F(A_1^*)$. By Lemma 1.2.4, F contains exactly 2 distinct subgroups isomorphic to Q_8 and these subgroup commute. Hence $C_F(A_1)$ and $C_F(A_1^*)$ must be these subgroups and so they commute.

Now suppose that $C_F(A_2) = C_F(A_1)$. Since $A_1A_2 = J$, as before we have that $C_F(A_1) = C_F(J) \leq C_F(A_3) = \langle t_\beta \rangle$, a contradiction. Hence $C_F(A_2) \neq C_F(A_1))$. A similar argument shows that $C_F(A_2) \neq C_F(A_1^*)$ and therefore $C_F(A_2) = \langle t_\beta \rangle$, completing the proof of (ii).

We have that $O_{3'}(C_{\widetilde{G}}(Y))$ is J-invariant since $J \leq N_{\widetilde{G}}(O_{3'}(C_{\widetilde{G}}(Y))$. Also, the group $O_{3'}(C_{\widetilde{G}}(Y)).2 \leq C_{\widetilde{G}}(Y)$ is not J-invariant. Therefore, $И^*_{C_{\widetilde{G}}(Y)}(J, 3') = \{O_{3'}(C_{\widetilde{G}}(Y))\}$. Since $A_i \geq Y$, we have that $C_{\widetilde{G}}(A_i) \leq C_{\widetilde{G}}(Y)$ and so any J-invariant subgroup of $C_{\widetilde{G}}(A_i)$ is also a J-invariant subgroup of $C_{\widetilde{G}}(Y)$. Therefore, any element of $И^*_{C_{\widetilde{G}}(A_i)}(J, 3')$ is contained in an element of $И^*_{C_{\widetilde{G}}(Y)}(J, 3')$. Since $J \leq C_{\widetilde{G}}(A_i)$, we have that $J \leq N_{\widetilde{G}}(O_{3'}(C_{\widetilde{G}}(A_i))$ and so $O_{3'}(C_{\widetilde{G}}(A_i))$ is J-invariant. Since in order to calculate $O_{3'}(C_{\widetilde{G}}(A_i))$ it has sufficed to consider the centralizer of A_i in the group $O_{3'}(C_{\widetilde{G}}) \in И^*_{C_{\widetilde{G}}(Y)}(J, 3')$, we see that $O_{3'}(C_{\widetilde{G}}(A_i)) \in И^*_{C_{\widetilde{G}}(A_i)}(J, 3')$. Therefore, by the uniqueness of $O_{3'}(C_{\widetilde{G}}(A_i))$, we have that $И^*_{C_{\widetilde{G}}(A_i)}(J, 3') = \{O_{3'}(C_{\widetilde{G}}(A_i))\}$. □

We note that since by Lemma 6.3.7 A_1 and A_1^* are conjugate, Lemma 6.3.8 implies that $И^*_{C_{\widetilde{G}}(A_1^*)}(J, 3') = \{O_{3'}(C_{\widetilde{G}}(A_1^*))\}$.

The following is a generalisation of Lemma 6.3.8.

Corollary 6.3.9 *Suppose that* $|J : A| = 3$.

(i) If A *is of type 1, then* $O_{3'}(C_{\widetilde{G}}(A_1)) \cong \mathrm{Q}_8$. *In addition, if* A *and* A' *are two distinct subgroups of type 1 that contain a given* Y, *then* $O_{3'}(C_G(A)) \neq O_{3'}(C_G(A')$ *and they commute.*

(ii) If A *is of types 2 or 3, then* $O_{3'}(C_{\widetilde{G}}(A)) \cong \langle t_\beta \rangle$.

Also, $И^*_{C_{\widetilde{G}}(A)}(J, 3') = \{O_{3'}(C_{\widetilde{G}}(A))\}$.

Proof. This follows directly from Lemma 6.3.8. □

Lemma 6.3.10 $\langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J : A| = 3 \rangle \cong 2_+^{1+8}$.

Proof. Let B be a subgroup of J of types 2 or 3. Then by Lemma 6.3.9, $O_{3'}(C_{\widetilde{G}}(B)) = \langle t_\beta \rangle$. If A is a subgroup of J of type 1, then $O_{3'}(C_{\widetilde{G}}(A)) \geq \langle t_\beta \rangle$. Hence

$$\langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J : A| = 3 \rangle = \langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J : A| = 3, A \text{ of type } 1 \rangle.$$

Suppose that A_i and A_j are two distinct subgroups of J of type 1. Then $A_i \cap A_j$ is equal to a subgroup of J conjugate to Y by Lemmas 3.1.3 and 6.2.10. So by Corollary 6.3.9, $O_{3'}(C_{\widetilde{G}}(A_i)) \cong O_{3'}(C_{\widetilde{G}}(A_j)) \cong \mathrm{Q}_8$. Also, $O_{3'}(C_{\widetilde{G}}(A_i)) \neq O_{3'}(C_{\widetilde{G}}(A_j))$ and they commute. Since there are four subgroups of type 1 by Lemma 6.3.3, we have four distinct subgroups isomorphic to Q_8 which commute. Therefore we can consider the central product of these subgroups and hence

$$\langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J : A| = 3 \rangle = \langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J : A| = 3, A \text{ of type } 1 \rangle \cong 2_+^{1+8}. \qquad \square$$

Corollary 6.3.11 *Suppose that $B \leq J$ such that $|B| = 3$ and $E = \langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J : A| = 3, A$ of type 1$\rangle$. If B is not $\widetilde{G}$-conjugate to Y or Z_β, then $C_E(B) = \langle t_\beta \rangle$.*

Proof. Since $J \geq B$, where B is a subgroup of type 3, we have that $C_E(J) \leq C_E(B) = \langle t_\beta \rangle$ by Lemma 6.3.6. By Lemma 6.3.10, E is an extra-special group, and so $Z(E) = \langle t_\beta \rangle$. Therefore $C_{E/Z(E)}(J) = \{0\}$. So by Lemma 1.1.5,

$$E/Z(E) = \bigoplus_{|J:A_i|=3} C_{E/Z(E)}(A_i).$$

Therefore, since if A_i is of type 2 or 3, $C_E(A_i) = \langle t_\beta \rangle$, we have that

$$E/Z(E) = \bigoplus_{A_i \text{ of type } 1} C_{E/Z(E)}(A_i).$$

Suppose that B is not conjugate to Z_β or Y. Then $A_i \not\geq B$ for all A_i of type 1. Therefore

$C_{Z/Z(E)}(B) = \{0\}$ and so $C_E(B) = \langle t_\beta \rangle$. $\square$

6.4 Completing the Proof of Theorem B

In this chapter we complete the proof of Theorem B which, with an application of a theorem due to Parrott [35], allows us to show that G is isomorphic to the Thompson sporadic simple group in our concluding remarks.

Throughout this chapter we use the notation defined and results proven earlier in this chapter and in Section 5.1.

Lemma 6.4.1 *$\langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J : A| = 3,\ A$ of type 1$\rangle$ is J-invariant. In particular, $\langle t_\beta \rangle \in И_{\widetilde{G}}(J, 3')$ but $\langle t_\beta \rangle \notin И^*_{\widetilde{G}}(J, 3')$.*

Proof. By Lemma 6.3.9, $O_{3'}(C_{\widetilde{G}}(A))$ is J-invariant for all A of type 1. Hence $\langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J : A| = 3,\ A$ of type 1$\rangle$ is J-invariant. Clearly $\langle t_\beta \rangle$ is J-invariant and thus $\langle t_\beta \rangle \in И_{\widetilde{G}}(J, 3')$. Since $\langle t_\beta \rangle < \langle O_{3'}(C_{\widetilde{G}}(A) \mid |J : A| = 3,\ A$ of type 1$\rangle$ it is clear that $\langle t_\beta \rangle \notin И^*_{\widetilde{G}}(J, 3')$. $\square$

We recall that $R \in И_{\widetilde{G}}(J, 3')$.

Lemma 6.4.2 *(i) $|И^*_{\widetilde{G}}(J, 3')| = 1$.*

(ii) R is extra-special of order 2^9.

(iii) $N_{\widetilde{G}}(J)$ acts irreducibly on $R/\langle t_\beta \rangle$.

Proof. Let $R_1 \in И^*_{\widetilde{G}}(J, 3')$. Then for $A \leq J$ with $|J : A| = 3$ as $C_{R_1}(A)$ is J-invariant,

$$C_{R_1}(A) \in И_{C_{\widetilde{G}}(A)}(J, 3') \subseteq И^*_{C_{\widetilde{G}}(A)}(J, 3') = \{O_{3'}(C_{\widetilde{G}}(A))\}$$

by Lemma 6.3.9, and so $C_{R_1}(A) \leq O_{3'}(C_{\widetilde{G}}(A))$. Therefore,

$$\begin{aligned} R &= \langle C_{R_1}(A) \mid |J:A| = 3\rangle \\ &\leq \langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J:A| = 3\rangle \\ &\cong 2_+^{1+8}, \end{aligned}$$

by Lemma 6.3.10. By Lemma 6.4.1, $\langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J:A| = 3,\ A \text{ of type } 1\rangle \in И_{\widetilde{G}}(J, 3')$. Therefore, by the maximality of R, this implies that $R_1 = R$ and (i) and (ii) hold.

We know that $N_{\widetilde{G}}(J)$ acts faithfully on $R/\langle t_\beta\rangle$. So $R/\langle t_\beta\rangle$ is a faithful $N_{\widetilde{G}}(J)$-module of dimension 8. So by Lemma 3.2.5, $R/\langle t_\beta\rangle$ is irreducible, and so $N_{\widetilde{G}}(J)$ acts irreducibly on $R/\langle t_\beta\rangle$. □

Lemma 6.4.3 *$O_{3'}(\widetilde{G}) \leq R$.*

Proof. By Coprime Action,

$$\begin{aligned} O_{3'}(\widetilde{G}) &= \langle C_{O_{3'}(\widetilde{G})}(A) \mid |J:A| = 3\rangle \\ &\leq \langle O_{3'}(C_{\widetilde{G}}(A)) \mid |J:A| = 3\rangle \\ &\cong 2_+^{1+8} \\ &= R, \end{aligned}$$

by Lemma 6.3.10 and the result follows. □

Since R is extra-special, $Z(R) = \langle t_\beta\rangle$, we have that $N_G(R) \leq N_G(Z(R)) = N_G(\langle t_\beta\rangle) = C_G(t_\beta) = \widetilde{G}$. Hence $N_G(R) = N_{\widetilde{G}}(R)$ and we may drop the $\widetilde{\ }$-notation.

Lemma 6.4.4 *$C_G(R) = \langle t_\beta\rangle$.*

Proof. We have that $C_G(R) = C_{\widetilde{G}}(R)$. If 3 divides $|C_{\widetilde{G}}(R)|$, then $J \leq C_{\widetilde{G}}(R)$ by Lemma 6.3.1. However, $C_J(R) = 1$ and so $C_G(R) = C_{\widetilde{G}}(R)$ is a $3'$-group. Also $C_G(R)$ is normalized by J. Since $И^*_{\widetilde{G}}(J, 3') = \{R\}$, and R is not abelian, we see that $C_G(R) = C_{\widetilde{G}}(R) = \langle t_\beta \rangle$. $\square$

Let $L = N_G(R)$. So, by Lemma 6.4.4, $L/C_G(R) = L/\langle t_\beta\rangle \hookrightarrow \mathrm{Aut}(2^{1+8}_+) = 2^8 : \mathrm{GO}^+_8(2)$ (see Theorem 1.2.8). Hence L/R is isomorphic to a subgroup of $\mathrm{Aut}(2^{1+8}_+)/\,\mathrm{Inn}(2^{1+8}_+) \cong \mathrm{GO}^+_8(2)$. Since, by Lemma 6.4.2, $N_{\widetilde{G}/\langle t_\beta\rangle}(J)$ acts irreducibly on $R/\langle t_\beta\rangle$ and $N_{\widetilde{G}/\langle t_\beta\rangle}(J) \cong 3^3 : \mathrm{Sym}(4)^+$ by Lemma 6.2.5, we see that $L/R \geq 3^3 : \mathrm{Sym}(4)^+$. In addition to this, the Sylow 3-subgroups of L are isomorphic to the Sylow 3-subgroups of $\mathrm{Alt}(9)$ by Lemma 6.1.14 and hence have order 3^4. We also note that $3^3 : \mathrm{Sym}(4)^+$ is maximal in $\mathrm{Alt}(9)$. Therefore, by [8, pages 46 and 85] we see that the maximal subgroups of $\mathrm{GO}^+_8(2)$ which have Sylow 3-subgroups of order at least 3^4 and contain a subgroup isomorphic to $3^3 : \mathrm{Sym}(4)^+$ are:

(*i*) a subgroup of $\mathrm{GO}^+_8(2)$ which contains $\mathrm{U}_4(2)$, such as $\mathrm{Sp}_6(2)$ or a subgroup of $(3 \times \mathrm{U}_4(2)) : 2$;

(*ii*) $\mathrm{Sym}(9)$;

(*iii*) $3^4 : 2^4.\,\mathrm{Sym}(4) \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(4)$.

Therefore, L/R must be isomorphic to a subgroup of one of these maximal subgroups of $\mathrm{O}^+_8(2)$.

We first eliminate case (*i*).

Lemma 6.4.5 *L/R does not contain a subgroup isomorphic to* $\mathrm{U}_4(2)$.

Proof. Suppose that L/R contains a subgroup isomorphic to $\mathrm{U}_4(2)$. We see from [8, page 26], this implies that $L/R \geq M \cong 3^{1+2} : 2^{\cdot}\,\mathrm{Alt}(4)$. So, since $Z_\beta \leq L$, we have that $Z_\beta R \leq M \cong 3^{1+2} : 2^{\cdot}\mathrm{Alt}(4)$ and $Z_\beta \in \mathrm{Syl}_3(Z_\beta R)$. Let $X = N_L(Z_\beta R)$. So, $X/R \cong 3^{1+2} : 2^{\cdot}\mathrm{Alt}(4)$, $R \trianglelefteq X$ and $Z_\beta R/R \leq X/R$. Hence, by the Frattini Lemma, $X = N_X(Z_\beta)Z_\beta R$ and therefore, $X/R = N_X(Z_\beta)R/R$. Since $N_X(Z_\beta) \leq \widetilde{G_\beta}$, we have that $X/R \leq \widetilde{G_\beta}R/R$. By Lemma 6.1.3, $\widetilde{G_\beta} = 2.3.(3_+^{1+2} \times \mathrm{Q}_8)$. Hence $|\widetilde{G_\beta}R/R| = 2.3^4$. However, as $|X/R| = 2^3 3^4 > |\widetilde{G_\beta}R/R|$, this gives rise to a contradiction. Hence L/R does not contain a subgroup isomorphic to $\mathrm{U}_4(2)$. $\square$

Lemma 6.4.6 *Suppose that L/R is isomorphic to a subgroup of $3^4 : 2^4.\mathrm{Sym}(4)$. Then $L/R \cong 3^3 : \mathrm{Sym}(4)^+$.*

Proof. Suppose that $F \cong 3^4 : 2^4.\mathrm{Sym}(4)$, $E = O_3(X)$ and $N = N_{\widetilde{G}/\langle t_\beta\rangle}(J)$. By Lemma 6.2.4, $N \cong 3^3 : \mathrm{Sym}(4)^+$. Let $H \cong L/R$. So we have that $N \leq H \leq F$ and $|H|_3 = |N|_3$. Then H normalizes E and so H normalizes $E \cap H$. Suppose that $E \leq H$. Since $|E| = |N|_3$, we have that N has elementary abelian Sylow 3-subgroups. This is a contradiction. Hence $E \not\leq H$ and $|E \cap H|_3 = 3^3$. Suppose that $E \cap H \neq J$. Then N normalizes $(E \cap H)J$. Since $|(E \cap H)J| \geq 3^3.3$, we have that $(E \cap H)J \in \mathrm{Syl}_3(H)$. Therefore N normalizes a Sylow 3-subgroup of H. We have that N contains a Sylow 3-subgroup of H and so N contains a normal Sylow 3-subgroup. However, this is a contradiction since N has more than one Sylow 3-subgroup. Therefore $E \cap H = J \trianglelefteq H$ and so $H = N$. $\square$

So we have that L/R is isomorphic to a subgroup of $\mathrm{Sym}(9)$. Therefore, using [8, page 37], we see that L/R is isomorphic to:

1. $3^3 : \mathrm{Sym}(4)^+$;
2. $\mathrm{Alt}(9)$;
3. $\mathrm{Sym}(3) \wr \mathrm{Sym}(3)$; or
4. $\mathrm{Sym}(9)$.

We show that we only need to consider cases 1 and 2.

Lemma 6.4.7 $L/R \ncong \mathrm{Sym}(3) \wr \mathrm{Sym}(3)$ *or* $\mathrm{Sym}(9)$.

Proof. By Lemma 6.2.4, $N_{\widetilde{G}/\langle t_\beta \rangle}(J) = 3^3 : \mathrm{Sym}(4)^+$. If $L/R \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(3)$ then $JR/R \trianglelefteq L/R$ and so $N_{\widetilde{G}/\langle t_\beta \rangle}(J) = \mathrm{Sym}(3) \wr \mathrm{Sym}(3)$. This is a contradiction and so L/R is not isomorphic to $\mathrm{Sym}(3) \wr \mathrm{Sym}(3)$. Since $\mathrm{Sym}(3) \wr \mathrm{Sym}(3) \leq \mathrm{Sym}(9)$, the other case follows immediately. $\square$

So $L/R = K$, where $K \cong 3^3 : \mathrm{Sym}(4)^+$, or $K \cong \mathrm{Alt}(9)$. We note that Lemma 6.2.5 implies that we can consider the cyclic subgroups of J of order 3 as being generated by elements in class 3A, 3B or 3C of $3^3 : \mathrm{Sym}(4)^+$ or $\mathrm{Alt}(9)$ as defined in Notation 3.2.1. For the rest of this section we use this notation, unless otherwise stated.

Lemma 6.4.8 *Z_β is generated by an element from K-conjugacy class 3B and Y is generated by an element from K-conjugacy class 3C.*

Proof. This follows from Lemma 6.2.10. $\square$

Lemma 6.4.9 *The elements of K-conjugacy class 3A act fixed-point-freely on $R/\langle t_\beta \rangle$.*

Proof. Let $a \in L$ have order 3 such that aR is a 3-cycle in K. Then $\langle a \rangle$ is not conjugate to Y or Z_β by Lemma 6.4.8. So by Corollary 6.3.11, $C_R(a) = \langle t_\beta \rangle$. Hence the result follows. $\square$

We note that Lemma 6.4.9 implies that we can use the results from Lemmas 3.2.4 and 3.2.6.

Lemma 6.4.10 *Let $L/R = K$, where $K \cong \mathrm{Alt}(9)$.*

(i) Suppose that zR is in K-conjugacy class 3B and $Z' = \langle z \rangle$. Then $N_{\widetilde{G}}(Z') \leq L$.

(*ii*) *Suppose that yR is in K-conjugacy class 3C and $Y' = \langle y \rangle$. Then $N_{\widetilde{G}}(Y') \leq L$.*

Proof. (*i*) We have that $N_L(Z') \leq N_{\widetilde{G}}(Z')$. By Lemma 6.4.8 Z' is L-conjugate Z_β. This implies that Z' is 3-central in G, and hence is $\widetilde{G}$-conjugate to Z_β and so $N_{\widetilde{G}}(Z') \cong \widetilde{G_\beta} \sim (3^{1+2} \times \mathrm{Q}_8).3.2$ by assumption and Lemma 6.1.3. Hence $|N_{\widetilde{G}}(Z')| = 2^4 3^4$. We have that $|C_L(Z')| = |C_R(Z')||C_K(Z'R)|$. Since Z' is generated by an element in K-conjugacy class 3B, by Lemma 3.2.6 (*iii*), $|C_R(Z')| = 2^3$. Also, by Lemma 3.2.2 (*iii*), we have $|C_K(Z'R)| = 3^4$. Therefore $|C_L(Z')| = 2^3 3^4$. Since $|N_L(Z') : C_L(Z')| = 2$ by Lemma 3.2.2 (*iii*), this implies that $|N_L(Z')| = 2^4 3^4$ and hence $N_L(Z') = N_{\widetilde{G}}(Z')$. Therefore $N_{\widetilde{G}}(Z') \leq L$ as required.

(*ii*) We have that $N_L(Y') \leq N_{\widetilde{G}}(Y')$. By Lemma 6.4.8, Y' is L-conjugate to Y. This implies that $C_G(Y') \cong C_G(Y)$ and so Y' is $\widetilde{G}$-conjugate to Y. So $|N_{\widetilde{G}}(Y')| = |N_G(Y)| = 2^7 3^3$ by Lemma 6.1.6. We have $|C_L(Y')| = |C_R(Y')||C_K(Y'R)|$. Since Y' is generated by an element in K-conjugacy class 3C, by Lemma 3.2.6 (*iii*), $|C_R(Y')| = 2^5$. Also, by Lemma 3.2.2 (*iii*), we have, $|C_K(Y'R)| = 54 = 2.3^3$. Therefore $|C_L(Y')| = 2^6 3^3$. Since $|N_L(Y') : C_L(Y')| = 2$ by Lemma 3.2.2 (*iii*), we have that $|N_L(Y')| = 2^7 3^3$. Hence $N_L(Y') = N_{\widetilde{G}}(Y')$ and therefore $N_{\widetilde{G}}(Y') \leq L$ as required. □

Theorem 6.4.11 *$\widetilde{G} = L$.*

Proof. For $L = N_G(R) = N_{\widetilde{G}}(R)$, where $R \cong 2_+^{1+8}$ we have two cases:

1. $L/R = K \cong \mathrm{Alt}(9)$; or

2. $L/R = K \cong 3^3 : \mathrm{Sym}(4)^+$,

where R is an extra-special group of order 2^9. By Lemma 6.4.9, the 3-cycles of K act fixed-point-freely on $R/\langle t_\beta \rangle$ in both cases. Clearly $L/\langle t_\beta \rangle \geq 2^8.3^3 : \mathrm{Sym}(4)^+$ and by Lemmas 6.2.5 and 6.4.2, $N_{\widetilde{G}/\langle t_\beta \rangle}(J) \cong 3^3 : \mathrm{Sym}(4)^+$ acts irreducibly on $R/\langle t_\beta \rangle$.

First suppose case 1 occurs. If $x \in L$ such that xR is in K-conjugacy class 3B or 3C, then $N_{\widetilde{G}}(\langle x \rangle) \leq L$ by Lemma 6.4.10. If x and y are not conjugate in L and xR and yR are in K-conjugacy classes 3A, 3B or 3C, then x and y are not $\widetilde{G}$-conjugate by Lemma 6.2.6. So Theorem 3.3.3 implies that $R/\langle t_\beta \rangle$ is strongly closed in $L/\langle t_\beta \rangle$ with respect to $\widetilde{G}$.

Now, suppose case 2 occurs. By Theorem 3.3.5, we see that $R/\langle t_\beta \rangle$ is strongly closed in $L/\langle t_\beta \rangle$ with respect to $\widetilde{G}/\langle t_\beta \rangle$.

In both cases, since $R/\langle t_\beta \rangle$ is strongly closed in $L/\langle t_\beta \rangle$ with respect to $\widetilde{G}/\langle t_\beta \rangle$ and hence $R/\langle t_\beta \rangle$ is strongly closed in $S/\langle t_\beta \rangle$ with respect to $\widetilde{G}/\langle t_\beta \rangle$ where $S \in \mathrm{Syl}_2(L/\langle t_\beta \rangle)$. Therefore $R/\langle t_\beta \rangle$ is weakly closed in $S/\langle t_\beta \rangle$ with respect to $\widetilde{G}/\langle t_\beta \rangle$ and so by Lemma 1.1.20, for $S \in \mathrm{Syl}_2(\widetilde{G}/\langle t_\beta \rangle)$. Therefore, Theorem 3.3.6 implies that

$$\widetilde{G}/\langle t_\beta \rangle = O_{2'}(\widetilde{G}/\langle t_\beta \rangle)L/\langle t_\beta \rangle.$$

Suppose that $X \trianglelefteq \widetilde{G}$ such that 3 divides $|X|$. Then by Lemma 6.3.1, $X \geq J\widetilde{G_\beta}$. Since $\widetilde{G_\beta}$ has even order, this implies that $X \not\leq O_{2'}(\widetilde{G})$. So, $|O_{2'}(\widetilde{G}/\langle t_\beta \rangle)|$ is not divisible by 3. However, $O_{3'}(\widetilde{G}/\langle t_\beta \rangle)$ is a 2-group by Lemma 6.4.3. Hence $O_{2'}(\widetilde{G}/\langle t_\beta \rangle) = 1$. Therefore $\widetilde{G}/\langle t_\beta \rangle = L/\langle t_\beta \rangle$ and so $\widetilde{G} = L$ as required. □

Theorem 6.4.12 *$\widetilde{G} \cong 2_+^{1+8}.\,\mathrm{Alt}(9)$.*

Proof. By Theorem 6.4.11, either $\widetilde{G} \cong 2_+^{1+8}.3^3 : \mathrm{Sym}(4)^+$, or $\widetilde{G} \cong 2_+^{1+8}.\,\mathrm{Alt}(9)$. By Lemma 6.1.7, $\widetilde{G}$ contains a subgroup isomorphic to $\mathrm{L}_2(8)$. Since $2_+^{1+8}.3^3 : \mathrm{Sym}(4)^+$ is soluble and $\mathrm{L}_2(8)$ is not, $\mathrm{L}_2(3) \not\leq 2_+^{1+8}.3^3 : \mathrm{Sym}(4)^+$. Therefore, the former case cannot occur and hence $\widetilde{G} \cong 2_+^{1+8}.\,\mathrm{Alt}(9)$ as required. □

Theorem 6.4.13 *$G \neq \widetilde{G}O_{2'}(G)$.*

Proof. Suppose $G = C_G(t)O_{2'}(G)$. Clearly G contains a subgroup isomorphic to $\mathrm{G}_2(3)$ as $G \geq N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$. Since 2 divides $|\mathrm{G}_2(3)|$, we have that $O_{2'}(G)$ does not contain a subgroup isomorphic to $\mathrm{G}_2(3)$. Hence $2_+^{1+8}.\,\mathrm{Alt}(9)$ contains a subgroup isomorphic to $\mathrm{G}_2(3)$. Therefore, as 13 divides $|\mathrm{G}_2(3)|$ and 13 does not divide $|2_+^{1+8}.\,\mathrm{Alt}(9)|$, this gives us a contradiction. Thus $G \neq C_G(t)O_{2'}(G)$. □

Concluding Remarks

We recall that throughout this thesis we have the following hypothesis.

Hypothesis A *Let G be a finite group and $S \in \mathrm{Syl}_3(G)$. Suppose that:*

(i) $Z_\beta = Z(S)$ *has order* 3 *and* $Z_\alpha = Z_2(S)$ *has order* 9*;*

(ii) $G_\alpha = N_G(Z_\alpha) \sim 3^{2+3+2+2} : 2.\mathrm{Sym}(4)$ *is* 3*-constrained;*

(iii) $G_\beta = N_G(Z_\beta) \sim 3^{1+2+1+2+1+2} : 2.\mathrm{Sym}(4)$ *is* 3*-constrained; and*

(iv) $O_3(\langle G_\alpha, G_\beta\rangle) = 1$.

We have proven the following results.

Theorem A *Suppose that G is a $\mathcal{K}$-proper group and that $S \in \mathrm{Syl}_3(G)$ such that G and S satisfy Hypothesis A. Then there exists a subgroup $Y \leq G$, such that $|Y| = 3$ and $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$.*

Proof. This is Theorem 5.3.3. $\square$

Theorem B *Let G and S satisfy Hypothesis A. Suppose that Y is the subgroup of order 3 in Theorem A and assume that $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$. Then there exists an involution $t \in G$ such that $G \neq C_G(t)O_{2'}(G)$ and $C_G(t)$ satisfies $R = O_2(C_G(t))$ is extra-special of order 2^9 and $C_G(t)/R \cong \mathrm{Alt}(9)$.*

Proof. We see from Theorem 6.4.13 that $G \neq C_G(t)O_{2'}(G)$ and the properties of $C_G(t)$ follow from Theorem 6.4.12. $\square$

Corollary *Suppose G, S and Y satisfy Theorem B. Then $G \cong \mathrm{Th}$.*

Proof. Since the conclusions of Theorem B are the hypotheses of Theorem 1.1.23, Parrott's Theorem, we have that $G \cong \mathrm{Th}$. $\square$

The proof of Theorem A relies heavily on a $\mathcal{K}$-group hypothesis in order to recognise the group $\mathrm{G}_2(3)$. An alternative way of recognising this group would be to consider a completion, H, of an amalgam of type $\mathrm{G}_2(3)$ and take an involution $s \in H$. Them, $C_H(s)$ can be found in a similar way to how $C_G(t)$ was found in the proof of Theorem A. We would then be able to say that $H \cong \mathrm{G}_2(3)$. This would then help to eliminate the $\mathcal{K}$-proper hypothesis in Theorem B and prove the following conjecture.

Conjecture *Suppose that G and S satisfy Hypothesis A. Then there exists an involution $t \in G$ such that $C_G(t)$ has shape $2_+^{1+8}.\mathrm{Alt}(9)$. In particular, $G \cong Th$.*

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